CANONICAL JOIN REPRESENTATIONS AND JOIN IRREDUCIBLE ELEMENTS OF GARSIDE SHADOWS IN COXETER GROUPS

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ABSTRACT. In this article, we establish some combinatorial properties of elements in Coxeter groups. Firstly, we generalise Reading's theorem on the canonical join representations of elements in finite Coxeter groups to all Coxeter groups.

Secondly, we show that for any element x in a Coxeter group W and root β in its inversion set $\Phi(x)$, the set of elements $y \in W$ satisfying $\Phi(y) \cap \Phi(x) = \{\beta\}$ is convex in the weak order and admits a unique minimal representative. This is strongly connected to determining the cone type of the inverse of x and leads to efficient computational methods to determine whether arbitrary elements of W have the same cone type. We also give a characterisation of the join-irreducible elements of two important classes of Garside shadows in Coxeter groups.

Finally, we apply our results to explicitly describe the set of ultra-low elements in various families of Coxeter groups, thereby verifying a conjecture of Parkinson and the author for these groups. We give explicit descriptions and counts of the join-irreducible elements of the minimal Garside shadow in these cases.

CONTENTS

1. Introduction

In Reading's seminal work on the Lattice Theory of the Poset of Regions of hyperplane arrangements [13, Chapters 9 & 10], the theory of *shards* is developed. Intuitively, shards are "pieces" of hyperplanes in certain hyperplane arrangements in bijection with the join-irreducible elements in the poset of regions. Using this very general set-up, Reading is able to describe some intricate details of the combinatorics of (mainly finite) Coxeter groups. Of particular interest in this article is Reading's theorem which gives an explicit description of the *canonical join representation* for each element in finite Coxeter groups.

Let us briefly fix some definitions and terminology in order to precisely state this result (detailed definitions are given in Section 2). Let L be a poset. A *join representation* for $x \in L$ is an identity of the form $x = \bigvee U$ for some $U \subseteq L$. A join representation $x = \bigvee U$ is *irredundant* if there does not exist a proper subset $U' \subset U$ such that $x = \bigvee U'$. For subsets U, V of L, denote $U << V$ if for every $u \in U$ there exists $v \in V$ such that $u \leq v$. The relation $U << V$ is called a *join-refinement* since if $U << V$ then $\bigvee U \leq \bigvee V$. We have the following key definition.

Definition 0.1. Let L be a poset and $x \in L$. An expression $x = \bigvee U$ is a *canonical join representation* if U is irredundant and if any other join representation $x = \bigvee V$ has $U << V$.

Let (W, S) be a finitely generated Coxeter system and let Φ be an associated root system. For $w \in W$ let $\Phi(w)$ be the inversion set of w. A basis of $\Phi(w)$ is the set $\Phi^1(w)$, called the *short inversions* of w. The *right weak order* on W is defined by $x \preceq y$ if and only if $\ell(y) = \ell(x) + \ell(x^{-1}y)$. Equivalently, in terms of reduced words, $x \preceq y$ if and only if there is a reduced word for y which begins with a reduced word for x. The right descent set of w is $D_R(w) = \{s \in S \mid ws \preceq w\}$ and $\Phi^R(w) = \{-w\alpha_s \mid s \in D_R(w)\}$ is the set of *right-descent roots* of w. Note that $\Phi^R(w) \subseteq \Phi^1(w)$.

An application of Reading's theory of shards to finite Coxeter groups is the following.

Theorem 0.1. [13, Theorem 10-3.9] Suppose W is a finite Coxeter group and $w \in W$. For each $\beta \in \Phi^R(w)$ there is a unique minimal length element $j_\beta \in \{v \mid v \preceq w, \beta \in \Phi(v)\}$. The canonical join representation of w is $w = \bigvee\{j_\beta \mid \beta \in \Phi^R(w)\}$.

Reading's proof of Theorem 0.1 relies on geometric properties of hyperplane arrangements associated with finite Coxeter groups (in particular *tightness* of the Coxeter arrangement, see [13, Section 10-2.1]); properties which generally fail in the infinite setting.

In this article, we give a uniform proof of this result; generalising Theorem 0.1 to arbitrary Coxeter groups.

Theorem 1. *Let $w \in W$. For each $\beta \in \Phi^R(w)$ there is a unique minimal length element $j_\beta \in \{v \mid v \preceq w, \beta \in \Phi(v)\}$. The canonical join representation of w is $w = \bigvee\{j_\beta \mid \beta \in \Phi^R(w)\}$.*

To prove Theorem 1 we first prove a slightly stronger statement than the first part of the theorem; for each $\beta \in \Phi^1(w)$ there is a unique minimal length element $j_\beta \in \{v \mid v \preceq w, \beta \in \Phi(v)\}$. Secondly, we utilise the technology of the *short inversion* poset recently developed by Dyer, Hohlweg, Fishel and Marks [9] in their study of low-elements and Shi-arrangements. This connection leads us into the second part of our work here on the join-irreducible elements of two important classes of Garside shadows.

Garside shadows were introduced in [6] in the authors' study of Garside families in Artin-Tits groups; Garside shadows being the *projections* of Garside families from Artin-Tits groups to their associated Coxeter groups. It turns out Garside shadows are intimately connected to automatic and biautomatic structures in Coxeter groups and have attracted significant interest in recent years (see [17], [21], [22], [19] and [24] for some examples).

Let us give a brief survey of the advances in this area. Brink and Howlett showed in [3] how to explicitly construct an automaton recognising the language of reduced words in Coxeter groups using the notion of *elementary roots* $\mathscr{E}$ and *elementary inversion sets* $\mathscr{E}(W)$; the latter which generalises the well known *Shi-arrangement* in the study of affine Weyl groups (see [25]) to arbitrary Coxeter groups. Hohlweg, Nadeau and Williams then established that automata recognising the language of reduced words can be constructed more generally using Garside shadows in [15].

In particular, an important class of Garside shadows, the *low elements* L (and more generally *m-low elements* L_m for $m \in \mathbb{N}$, see [10]) have recently been extensively studied and have been shown to be in some sense the canonical representatives of elementary inversion sets $\mathscr{E}(W)$ (resp. m-elementary inversion sets $\mathscr{E}_m(W)$) (see [16], [5] and [9]).

Hohlweg, Nadeau and Williams work in [15] not only gave a new method of constructing automata but established some important conjectures as well. In particular, they conjectured that the automaton constructed from the smallest Garside shadow $\widetilde{S}$ is the minimal automaton (in terms of number of states) recognising the language of reduced words of W [15, Conjecture 2].

Motivated by this conjecture, in [21] we studied the minimal automaton via the study of the set of *cone types* $\mathbb{T}$ of W. In [21], we introduced the notion of a *regular partition* $\mathscr{R}$; a class of partitions which characterises the (reduced) automata recognising the language of reduced words of W. Furthermore, we showed that each part Q of the regular partition corresponding to the cone type automaton, $\mathscr{T}$ (called the *cone type arrangement* or *cone type partition*) contains a unique minimal length element g (called the *gate* of Q) which is a prefix of all elements in Q. The set of these minimal length cone type representatives Γ, called the *gates* of the cone type arrangement were conjectured to be the smallest Garside shadow [21, Conjecture 1].

In addition, we conjectured that there is a geometric characterisation of the elements of Γ in terms of their short inversions and their cone type [21, Conjecture 2]; in this article, we call this the *ultra-low conjecture*. This motivated the introduction of a new set of elements analogous to the low-elements called the *ultra-low* elements $\mathscr{U}$. Then the ultra-low conjecture is equivalent to the sets Γ and $\mathscr{U}$ being equal.

In [22], we verified that the set Γ in closed under join (see [22, Theorem 1.4]) and it was shown that Γ is closed under suffix in [21]; thus Γ is the smallest Garside shadow, verifying [21, Conjecture 1]. However, whether the set $\Gamma = \mathscr{U}$ remains a seemingly challenging open question and hence motivates much of the work in the third part of our note here.

Indeed, one of the contributions of the work in this article is resolving the ultra-low conjecture for rank 3 irreducible Coxeter groups, right-angled Coxeter groups and those whose Coxeter graph Γ_W is a complete graph, potentially simplifying the argument for a general resolution of the conjecture by reducing it to the rank 3 case (see Section 7).

Further motivation for our study of cone types is inspired by Osajda and Przytycki's recent breakthrough work in [19] showing that all Coxeter groups are biautomatic, finally confirming a long held conjecture. It was also recently shown that any Garside shadow can be used to construct a biautomatic structure (see [24]) hence increasing the importance of the understanding of cone types.

In Section 5 onwards, we study important subsets of the Garside shadows Γ and (to a lesser extent) L_m, which we call the *tight gates* of the cone type arrangement $\mathscr{T}$ and m-Shi arrangements $\mathscr{S}_m$ respectively. We denote these elements by Γ^0 and L_m^0 respectively. Let us fix additional terminology and notation before outlining our remaining results.

An element $x \in W$ is a *suffix* of y if $\ell(y) = \ell(yx^{-1}) + \ell(x)$. Like the notion of *prefix*, x is a suffix of y if there is a reduced word for y which ends with a reduced word for x. For $w \in W$ we denote

$$T(w) = \{v \in W \mid \ell(wv) = \ell(w) + \ell(v)\}$$

to be the *cone type* of w. The set of cone types of W is $\mathbb{T} = \{T(w) \mid w \in W\}$. For $T \in \mathbb{T}$ the *projection* of the cone type T (also called the *cone type part*) is the set

$$Q(T) = \{x \in W \mid T(x^{-1}) = T\}$$

The set of *boundary roots* of T is

$$\partial T = \{\beta \in \Phi^+ \mid \exists\, y \in W \text{ with } \Phi(w) \cap \Phi(y) = \{\beta\} \text{ for some } w \in Q(T)\}$$

The boundary roots ∂T is the most precise set of roots which determines the cone type T [21, Theorem 2.8]. We say a root $\beta \in \Phi^+$ is *super-elementary* if there exists $x, y \in W$ with $\Phi(x) \cap \Phi(y) = \{\beta\}$ and denote the set of super-elementary roots by $\mathcal{S}$. In the course of studying the set Γ^0, we also examine elements which are *witnesses* of boundary roots of cone types.

Definition 1.1. Let $x \in W$ and $\beta \in \Phi$. An element $y \in W$ is a *witness of* β *with respect to* x if

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

We denote $\partial T(x^{-1})_\beta$ to be the set of *witnesses of* β *with respect to* x. More precisely,

$$\partial T(x^{-1})_\beta = \{y \in W \mid \Phi(x) \cap \Phi(y) = \{\beta\}\}$$

Remark 1.1. For each cone type T and $x, y \in Q(T)$ it is not generally true that $\partial T(x^{-1})_\beta = \partial T(y^{-1})_\beta$ as illustrated in the following figure for a cone type in the Coxeter group of type $\widetilde{G}_2$.

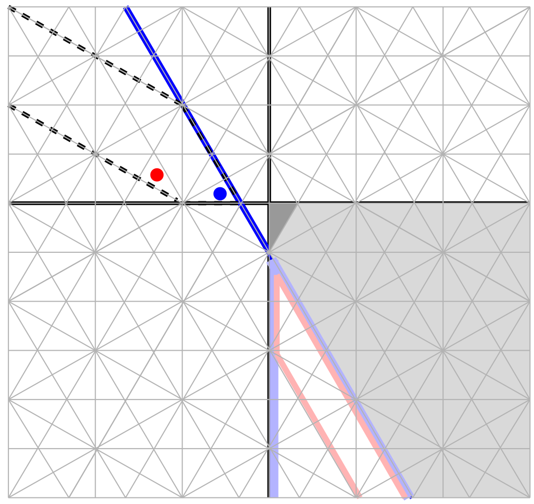

FIGURE 1. Let x and y be the elements represented by the blue and red dots respectively. The cone type $T := T(x^{-1}) = T(y^{-1})$ is represented by the gray shaded region. Let $\beta \in \partial T$ correspond to the blue hyperplane. The black hyperplanes correspond to the remaining boundary roots of T. The region bounded by the dotted lines is $Q(T)$. The region bounded by the red highlighted lines is $\partial T(y^{-1})_\beta$ and the region bounded by the blue highlighted lines is $\partial T(x^{-1})_\beta$.

Definition 1.2. Let $T \in \mathbb{T}$ be a cone type and $\beta \in \partial T$. The *witnesses of β with respect to T* is

$$\partial T_\beta = \{y \in W \mid \Phi(x) \cap \Phi(y) = \{\beta\} \text{ for all } x \in Q(T)\}$$

For a subset $X \subseteq W$, we say X is *gated* if there is a unique minimal length element $x \in X$ such that $x \preceq y$ for all $y \in X$. We say the subset X is *convex* if for all $x, y \in X$ and all reduced expressions $x^{-1}y = s_1 \cdots s_n$, the elements $xs_1 \cdots s_j$ with $0 \le j \le n$ is in X. In terms of the Cayley graph X^1 of W this means that all elements on geodesics between x and y are in X.

The second main result in this article is the following (a consequence of Lemma 4.2 and Theorem 4.1).

Theorem 2. *For $x \in W$ and $\beta \in \partial T(x^{-1})$ there exists a unique minimal length element $y \in W$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

Furthermore, $y \in \Gamma^0$ and the set $\partial T(x^{-1})_\beta$ is a convex, gated set in W.

In [21] we showed that for each boundary root $\beta \in \partial T$ and $x \in W$ with $T(x^{-1}) = T$ there is *a* witness $y \in W$ such that $y \in \Gamma$ and $\Phi^R(y) = \{\beta\}$ (see [21, Proposition 4.36]). However, it was not known whether such an element y is unique, let alone that it is the minimal length representative of $\partial T(x^{-1})_\beta$ (and in fact it is the minimal length representative of ∂T_β as well). As a consequence of Theorem 2 we obtain the following, which by the uniqueness of the element y, is a stronger statement than [21, Theorem 2.6].

Corollary 1. *For each $T \in \mathbb{T}$ there is a unique minimal length element $y \in \Gamma^0$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

for all $x \in Q(T)$. Hence, y is the gate of the sets $\partial T(x^{-1})_\beta$ for all $x \in Q(T)$ and the set ∂T_β.

Theorem 2 reveals that each cone type T is associated with a number of convex, gated subsets of W, including its cone type part $Q(T) \in \mathscr{T}$ and the sets $\partial T(x^{-1})_\beta$ for each $x \in Q(T)$ and $\beta \in \partial T$ (as well as the cone type T itself with the identity as the gate).

We illustrate in Figure 2 an example in the case of the Coxeter group of type $\widetilde{G}_2$. The figure shows a cone type T, its cone type part $Q(T)$ and the sets $\partial T(x^{-1})_\beta$ for each $\beta \in \partial T$ for the minimal element x of $Q(T)$.

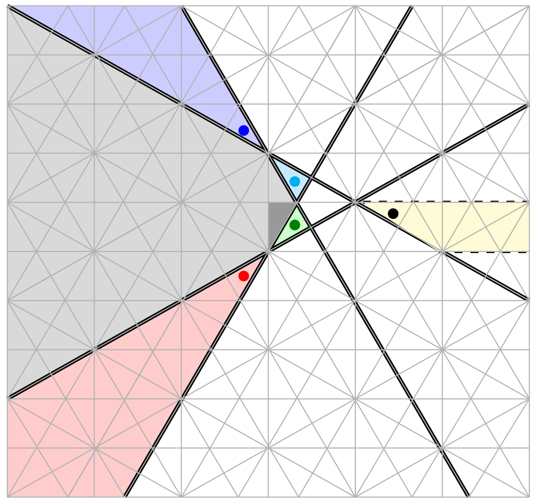

FIGURE 2. Let the element x be represented by the alcove with the black dot. The grey shaded region is the cone type $T := T(x^{-1})$ (the darker alcove represents the identity). The elements w in the yellow shaded region are the elements such that $T(w^{-1}) = T$ (i.e. the cone type part $Q(T)$ of $\mathscr{T}$ corresponding to T). The remaining coloured shaded regions are the sets $\partial T(x^{-1})_\beta$ for each $\beta \in \partial T$ and the corresponding coloured dots are the gates of those regions.

Let us now describe precisely what we mean by *tight gates*. For a partition $\mathscr{P}$ of W we say $\mathscr{P}$ is *gated* if each part P of $\mathscr{P}$ is gated. We say $\mathscr{P}$ is *convex* if each part P is convex. If a partition $\mathscr{P}$ is gated with set of gates $\mathcal{X}$ we define

$$\mathcal{X}^0 = \{x \in \mathcal{X} \mid |\Phi^R(x)| = 1\}$$

to be the set of *tight gates* of $\mathscr{P}$ (or X). It follows by [21] and [9] that the cone type partition $\mathscr{T}$ and the m-Shi partitions $\mathscr{S}_m$ are convex, gated (and *regular*) partitions with gates Γ and L_m respectively (see Section 2.1.4 for more about Garside shadows and Garside partitions).

As an application of Theorem 2 we obtain a correspondence between the tight gates Γ^0 and the set of super-elementary roots $\mathcal{S}$ for a few classes of Coxeter groups, giving a precise size and description of Γ^0 in these cases. Other main results in this work show that the sets L_m^0 for $m \in \mathbb{N}$ and Γ^0 are the most fundamental elements of L_m and Γ in the following sense (see Theorem 5.2 and Theorem 5.3).

Theorem 3. *Let L_m be the gates of the m-Shi partition $\mathscr{S}_m$. Then*

$$L_m^0 = \{g \in L_m \mid |\Phi^R(g)| = 1\}$$

is the set of join-irreducible elements of the partially ordered set $(L_m, \preceq)$.

Theorem 4. *Let Γ be the gates of the cone type partition $\mathscr{T}$. Then*

$$\Gamma^0 = \{g \in \Gamma \mid |\Phi^R(g)| = 1\}$$

is the set of join-irreducible elements of the partially ordered set $(\Gamma, \preceq)$.

A consequence of Theorem 4 is that the set of tight gates Γ^0 completely determines the cone type arrangement (see Corollary 5.2). In addition, we obtain the following (see Corollary 4.5).

Corollary 2. *Let G be a Garside shadow. The set*

$$G^0 = \{g \in G \mid |\Phi^R(g)| = 1\}$$

is closed under suffix.

Corollary 2 then leads to a very efficient algorithm to compute the set Γ^0 and $\mathcal{S}$ without having to compute the entire set Γ first (and similarly for the sets L_m^0). A consequence of this result is that to determine whether two elements x, y have the same cone type, one is only required to compute a set strictly smaller than Γ (with only a few exceptions). Our computations using Sagemath ([7]) show that this set is in many cases much smaller than Γ as the rank of W grows (see Algorithm 6.1 and Figure 5).

In Section 7 we prove the *ultra-low conjecture* [21, Conjecture 2] for a few selected classes of Coxeter groups (see Theorem 7.1 and Section 7.2).

Theorem 5. *Let W be a Coxeter group of one of the following types*

(i) W is finite dihedral,
(ii) W is irreducible rank 3,
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then $\mathscr{U} = \Gamma = \widetilde{S}$.

The following is then a consequence of Corollary 8.1 and Corollary 8.2.

Theorem 6. *Let (W, S) be a Coxeter group of one of the following types*

(i) W is finite dihedral,
(ii) W is irreducible rank 3,
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then for each non-simple super-elementary root β there is a unique pair of tight gates x, y such that

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

and $\Phi^R(x) = \Phi^R(y) = \{\beta\}$. Furthermore,

$$|\Gamma^0| = 2|\mathscr{E}| - |S|$$

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2. Preliminaries

We recall the necessary facts on Coxeter systems, Cayley graphs, root systems, Garside shadows, cone types, gates, short inversion poset and ultra-low elements. For general Coxeter group theory the primary references are [18], [2] and [23]. For background on Garside shadows and short inversions our primary references are [16], [14], [9] and for cone types and ultra-low elements see [21].

2.1. Coxeter Groups.

Let (W, S) be a Coxeter system with $|S| < \infty$. The *length* of $w \in W$ is

$$\ell(w) = \min\{n \geq 0 \mid w = s_1 \dots s_n \text{ with } s_1, \dots, s_n \in S\},$$

and an expression $w = s_1 \dots s_n$ with $n = \ell(w)$ is called a *reduced expression* or *reduced word* for w. The *left descent set* of w is $D_L(w) = \{s \in S \mid \ell(sw) = \ell(w) - 1\}$, similarly the *right descent set* is $D_R(w) = \{s \in S \mid \ell(ws) = \ell(w) - 1\}$. It is well known that the set

$$R = \{wsw^{-1} \mid w \in W, s \in S\}$$

is the set of *reflections* of W (see [2, Chapter 1]). For $J \subseteq S$, we denote $W_J \leq W$ to be the standard parabolic subgroup of W. For each $w \in W$ there is a unique reduced decomposition $w = w_J \cdot w^J$ where $w_J \in W_J$ and $w^J \in W^J$, with $W^J = \{v \in W \mid D_L(v) \cap J = \emptyset\}$.

2.1.1. *Root System of W.* Let V be an $\mathbb{R}$-vector space with basis $\{\alpha_s \mid s \in S\}$. Define a symmetric bilinear form on V by linearly extending $\langle \alpha_s, \alpha_t \rangle = -\cos(\pi/m(s,t))$. The Coxeter group W acts on V by the rule $sv = v - 2\langle v, \alpha_s \rangle \alpha_s$ for $s \in S$ and $v \in V$, and the root system of W is $\Phi = \{w\alpha_s \mid w \in W, s \in S\}$. The elements of Φ are called *roots*, and the *simple roots* are $\Delta = \{\alpha_s \mid s \in S\}$.

Each root $\alpha \in \Phi$ can be written as $\alpha = \sum_{s \in S} c_s \alpha_s$ with either $c_s \geq 0$ for all $s \in S$, or $c_s \leq 0$ for all $s \in S$. In the first case α is called *positive* (written $\alpha > 0$), and in the second case α is called *negative* (written $\alpha < 0$). For $\beta \in \Phi$ we denote the coefficient of the simple root α_s by $\text{Coeff}_{\alpha_s}(\beta)$. The *support* of α is the set $J(\alpha) = \{s \in S \mid c_s \neq 0\}$. We denote the subgraph of the Coxeter graph Γ_W corresponding to $J(\alpha)$ by $\Gamma(\alpha)$. We say a root α has *non-spherical* support if $J(\alpha)$ generates an infinite Coxeter group. Denote Φ^+

(resp. Φ^-) to be the positive roots (resp. negative roots) of Φ. By definition of Φ, each root β can be written $\beta = w\alpha_s$ for some $s \in S$. Then the reflection $s_\beta = wsw^{-1}$ separates W into two sets (called *half spaces*)

$$H_\beta^- = \{w \in W \mid \ell(s_\beta w) < \ell(w)\} \text{ and } H_\beta^+ = \{w \in W \mid \ell(s_\beta w) > \ell(w)\}$$

Half spaces provide an important characterisation of convexity in W.

Lemma 2.1. [1, Proposition 3.94] *A subset $A \subseteq W$ is convex if and only if it is an intersection of half spaces.*

For $A \subset \Phi^+$, $cone(A)$ is the set of non-negative linear combinations of roots in A and $cone_\Phi(A) = cone(A) \cap \Phi^+$. The *(left) inversion set* of $w \in W$ is

$$\Phi(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) < 0\}.$$

An important subset of an inversion set $\Phi(w)$ is the *short inversions* $\Phi^1(w)$ (also called the *base* of $\Phi(w)$). These roots are used to define *low* elements and *ultra-low* elements (see Section 2.1.2 and Section 2.1.3).

Definition 2.2. [16, Proposition 4.6] Let $w \in W$. The *short inversions* of $\Phi(w)$ (also called *base* of $\Phi(w)$) is the set

$$\Phi^1(w) = \{\beta \in \Phi(w) \mid \ell(s_\beta w) = \ell(w) - 1\}$$

The set $\Phi^1(w)$ determines the inversion set $\Phi(w)$ in the following way (see [8, Lemma 1.7] and [14, Corollary 2.13]).

Proposition 2.1. *For $w \in W$ we have $\Phi(w) = cone_\Phi(\Phi^1(w))$ and if $A \subset \Phi^+$ is such that $\Phi(w) = cone_\Phi(A)$ then $\Phi^1(w) \subseteq A$.*

The following result will be used to compute $\Phi^1(sw)$ from $\Phi^1(w)$ whenever $\ell(sw) > \ell(w)$.

Proposition 2.2. [16, Theorem 4.10(2)] *Let $s \in S$ and $w \in W$. If $\ell(sw) > \ell(w)$ then*

$$\Phi^1(sw) = \{\alpha_s\} \sqcup s\big(\{\beta \in \Phi^1(w) \mid s_\beta w < ss_\beta w\}\big)$$

As noted earlier, another important subset of $\Phi(w)$ is the following.

Definition 2.3. Let $w \in W$. The set of *right-descent roots* of w (also called *final roots*) is the set

$$\Phi^R(w) = \{-w\alpha_s \mid s \in D_R(w)\}$$

A root $\beta \in \Phi^R(w)$ if and only if $s_\beta w = ws$ for some $s \in D_R(w)$. In terms of the Cayley graph X^1 the right descent roots correspond to the set of *final* or *last* walls crossed for geodesics from the identity e to w.

We collect some further useful facts regarding joins in the right weak order.

Proposition 2.3. [16, Proposition 2.8] *If $X \subset W$ is bounded (in the right weak order), then*

$$\Phi(\bigvee X) = cone_\Phi(\bigcup_{x \in X} \Phi(x))$$

Corollary 2.1. [16, Corollary 4.7 (2)] *If $x \vee y$ exists and $\ell(s_\beta(x \vee y)) = \ell(x \vee y) - 1$, then either $\ell(s_\beta x) = \ell(x) - 1$ or $\ell(s_\beta y) = \ell(y) - 1$.*

2.1.2. *m-elementary roots, m-low elements and the m-Shi partition.* We recall the notion of *dominance* of positive roots. A root $\beta \in \Phi^+$ *dominates* $\alpha \in \Phi^+$ if whenever $w^{-1}\beta < 0$ we have $w^{-1}\alpha < 0$. The set of *m-elementary roots* $\mathscr{E}_m$ is the set of positive roots which dominate m roots (other than itself). For $w \in W$, the *m-elementary inversion set* of w is the set $\mathscr{E}_m(w) = \Phi(w) \cap \mathscr{E}_m$. An element $w \in W$ is an *m-low element* if $\Phi^1(w) \subset \mathscr{E}_m$ or equivalently $\Phi(w) = cone_\Phi(\mathscr{E}_m(w))$. The set of *$m$-low elements* is denoted L_m (See [10] for more details).

We note that for the case $m = 0$, the mention of m and the subscript is often omitted. The following result of Brink provides very useful information about the elementary roots in terms of their support.

Lemma 2.4. [4, Lemma 4.1] *Suppose that $\alpha \in \Phi^+$ and $\Gamma(\alpha)$ contains a curcuit or an infinite bond. Then $\alpha \notin \mathscr{E}$.*

For all $m \in \mathbb{N}$, Fu ([11]) showed that $\mathscr{E}_m$ is finite and thus there are only finitely many sets $\mathscr{E}_m(w)$ for any Coxeter group W. The sets $\mathscr{E}_m(w)$ thus partition W into finitely many parts. We call the partition of W with parts $\mathscr{E}_m(w)$ the *m-Shi partition* (or *m-Shi arrangement*), denoted $\mathscr{S}_m$.

A recent result of Dyer, Fishel, Hohlweg and Marks shows that the set L_m is the set of unique minimal length representatives for each part of the *m-Shi* partition (see [9, Theorem 1.1]). In the language of [21], the m-Shi partition is a gated, convex partition with gates L_m (in fact, these partitions are also *regular*, which is further discussed in Section 2.1.6). Furthermore, Dyer and Hohlweg previously showed that L is a Garside shadow ([16]), which was then extended to all $m \in \mathbb{N}$ by Dyer ([10]). These results for the $m = 0$ case were also independently proven in [22]. The *tight gates* of $\mathscr{S}_m$ are

$$L_m^0 = \{x \in L_m \mid |\Phi^R(x)| = 1\}$$

2.1.3. *Super-elementary roots and ultra-low elements.*

Definition 2.5. A root $\beta \in \Phi^+$ is *super-elementary* if there exists $x, y \in W$ with

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

We let $\mathcal{S}$ denote the set of super-elementary roots. By the following result we have that $\mathcal{S} \subseteq \mathscr{E}$ and hence the set $\mathcal{S}$ is finite for every finitely generated Coxeter group. For a number of classes of Coxeter groups, including those of affine type, we have $\mathcal{S} = \mathscr{E}$ (see [21, Section 7]).

Lemma 2.6. [21, Lemma 2.4] *Let $x, y \in W$ and $\beta \in \Phi^+$. Suppose that $\Phi(x) \cap \Phi(y) = \{\beta\}$. Then:*

(1) $\beta \in \mathscr{E}$, and;
(2) $\beta \in \Phi^1(x) \cap \Phi^1(y)$

Definition 2.7. An element $w \in W$ is *ultra-low* if for each $\beta \in \Phi^1(x)$ there exists $y \in W$ with

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

We denote the set of all ultra-low elements by $\mathscr{U}$.

2.1.4. *Garside shadows and Garside partitions.* We briefly recall the basics of Garside shadows (for further details see [15]). A subset $B \subseteq W$ is a *Garside shadow* if B contains S and

(1) B is closed under join in the (right) weak order: if $X \subseteq B$ is bounded then $\bigvee X \in B$.
(2) B is closed under taking suffixes. If $w \in B$ then any suffix of w is in B.

For every Coxeter system (W, S) with S finite, the smallest Garside shadow $\widetilde{S} := Gar_S(S)$ in W is finite ([16, Corollary 1.2]). The smallest Garside shadow $\widetilde{S}$ was shown to be Γ (see [22, Theorem 1.4]). For an element $x \in W$ and Garside shadow G, denote $\pi_G(x)$ to be the longest prefix of x contained in G (this is called the *G-projection* of x).

The G-projection map induces a partition $\mathscr{G}$ of W with parts P_g for $g \in G$ where elements $x, y \in W$ are in P_g if and only if $\pi_G(x) = \pi_G(y) = g$. Since Γ and L_m are Garside shadows, the partition obtained in this way are the partitions $\mathscr{T}$ and $\mathscr{S}_m$ respectively (see [21]).

2.1.5. *Cone Types and Boundary roots of W.* The definition of cone types, cone type parts and boundary roots were introduced in Section 1. We recall some key properties related to these notions that will be useful for our work here.

The *cone type arrangement* or *cone type partition* $\mathscr{T}$ is the partition of W into the sets

$$Q(T) = \{x \in W \mid T(x^{-1}) = T\}$$

for each $T \in \mathbb{T}$. The partition $\mathscr{T}$ is the minimal *regular partition* of W as in discussed in some depth in [21]. The parts $Q(T)$ are in bijection with the cone types. The following theorem is an interesting characterisation of $\mathscr{T}$ and reveals that the parts $\{Q(T) \mid T \in \mathbb{T}\}$ of $\mathscr{T}$ can realised as intersections of the set of cone types $\mathbb{T}$.

Theorem 2.1. [26, Theorem 3.4.3] *For $x, y \in W$ define $x \sim y$ if and only if*

$$x \in T \text{ if and only if } y \in T \text{ for all } T \in \mathbb{T}$$

Then the partition of W by the $\sim$ equivalence classes is $\mathscr{T}$.

Each part $Q(T)$ contains a unique minimal length representative g_T, called the *gate* of $Q(T)$ which is a prefix of all elements in $Q(T)$.

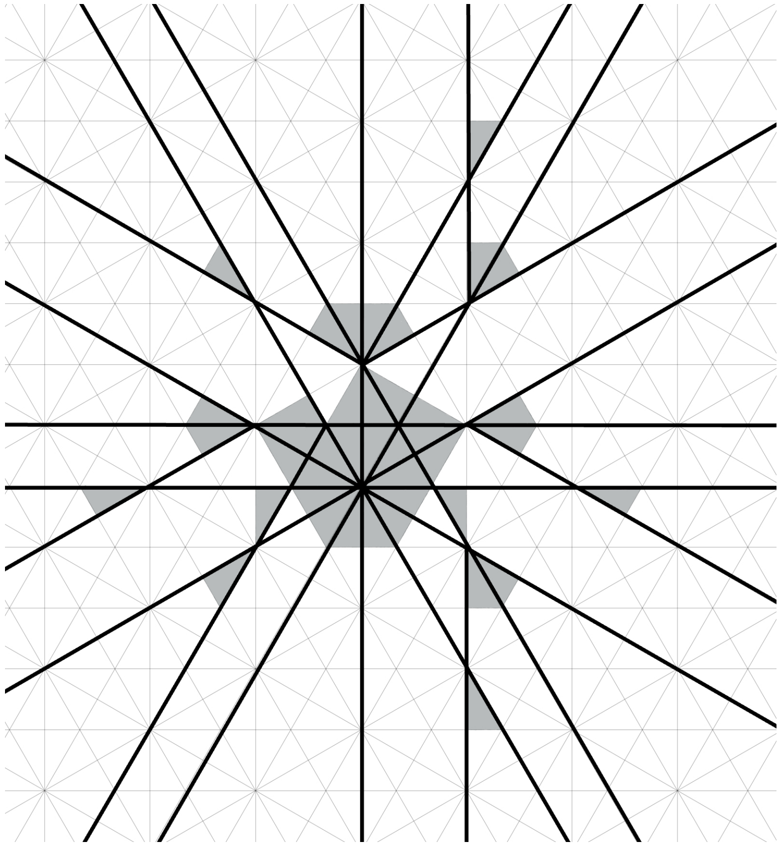


FIGURE 3. The cone type partition $\mathscr{T}$ for the Coxeter group of type $\widetilde{G}_2$. For each cone type T, the part $Q(T)$ contains the elements x such that $T(x^{-1}) = T$. The shaded in alcove of each part represents the inverse of the unique minimal length cone type representatives m_T^{-1}. These are the *gates* of $\mathscr{T}$.

Theorem 2.2. [21, Theorem 1] *For each cone type T there is a unique element $m_T \in W$ of minimal length such that $T(m_T) = T$. Moreover, if $w \in W$ with $T(w) = T$ then m_T is a suffix of w.*

By Theorem 2.2 and as illustrated in Figure 3 from the point of view of the cone type partition $\mathscr{T}$, for $x \in W$, it is in some sense more natural for one to consider the cone type $T(x^{-1})$. The *gates of* $\mathscr{T}$ is the set

$$\Gamma = \{m_T^{-1} \mid T \in \mathbb{T}\}$$

and the *tight gates* of $\mathscr{T}$ is the set

$$\Gamma^0 = \{g \in \Gamma \mid |\Phi^R(g)| = 1\}$$

The following theorem gives a characterisation of a gate x in terms of $\Phi^R(x)$.

Theorem 2.3. [21, Theorem 4.33] *Let $x \in W$. Then $x \in \Gamma$ if and only if for each $\beta \in \Phi^R(x)$ there exists $w \in W$ with $\Phi(x) \cap \Phi(w) = \{\beta\}$.*

Theorem 2.4. [21, Theorem 2.6] *Let T be a cone type. If $T = T(x^{-1})$ then $\beta \in \partial T$ if and only if there exists $w \in W$ with*

$$\Phi(x) \cap \Phi(w) = \{\beta\}$$

Moreover, if $\beta \in \partial T$ then there exists $w \in W$, independent of x, such that $\Phi(x) \cap \Phi(w) = \{\beta\}$ whenever $T = T(x^{-1})$.

We list some further relevant results regarding cone types and their boundary roots.

Lemma 2.8. [21, Lemma 1.15] *If $x \preceq y$ then $T(y^{-1}) \subseteq T(x^{-1})$.*

Theorem 2.5. [21, Corollary 2.9] *Let T be a cone type. If $T = T(x^{-1})$ Then*

$$T = \bigcap_{\Phi(x)} H_\alpha^+ = \bigcap_{\mathscr{E}(x)} H_\alpha^+ = \bigcap_{\Phi^1(x)} H_\alpha^+ = \bigcap_{\Phi^1(x)\cap\mathscr{E}(x)} H_\alpha^+ = \bigcap_{\partial T(x^{-1})} H_\alpha^+$$

Proposition 2.4. [21, Proposition 1.19] *If $X \subseteq W$ is bounded with $y = \bigvee X$ then $T(y^{-1}) = \bigcap_{x\in X} T(x^{-1})$.*

Proposition 2.5. [26, Proposition 3.2.4] *Let T be a cone type.*

(i) *If $\beta \in \Phi^+$ is a boundary root of T then there exists $w \in W$ such that $\Phi(x) \cap \Phi(w) = \{\beta\}$ for all $x \in W$ with $T(x^{-1}) = T$.*
(ii) *For $x, w \in W$, if $T(x^{-1}) = T$ and $\Phi(x) \cap \Phi(w) = \{\beta\}$ then $\beta \in \partial T$.*

As noted in the introduction, the following result is known.

Proposition 2.6. [21, Proposition 4.34] *Let $x \in W$ and $\beta \in \Phi^+$. Suppose there exists $w \in W$ such that $\Phi(x) \cap \Phi(w) = \{\beta\}$, and let w be of minimal length subject to this property. Then $\Phi^R(w) = \{\beta\}$ and w is a gate.*

The following is then a consequence of Theorem 2.4 above and the definition of the minimal length cone type representatives, low-elements and ultra-low elements.

Proposition 2.7. [21, Proposition 6.2] *For each Coxeter group W,*

$$\mathscr{U} \subseteq \Gamma \subseteq L$$

2.1.6. *Automata and Regular Partitions.* It will be helpful to briefly mention the connection between partitions of W and automata for recognising the language of reduced words of (W, S).

For details of automata we refer the reader to [21, Section 1.8]. In that article, we characterise (the states of) automata recognising the language of reduced words of (W, S) in terms of the notion of a *regular partition* (see [21, Theorem 2]). In brief, each automaton arises from a regular partition and for each regular partition there exists an explicitly defined automaton. Primary examples of regular partitions are the m-Shi partitions, and the cone type partition $\mathscr{T}$. In addition, by the results of [17] and [21], any Garside shadow G gives rise to a regular partition via the G-projection map as described in Section 2.1.4.

2.1.7. *Maximal dihedral reflection subgroups.* We briefly recall the notion of maximal dihedral reflection subgroups as this is required for the definition of the short inversion poset. For a more in-depth discussion see [16, Section 2.5]. A dihedral reflection subgroup W' of W is a subgroup of rank 2. Every dihedral reflection subgroup is contained in a *maximal* dihedral reflection subgroup.

For $\alpha, \beta \in \Phi^+$, we denote $\mathcal{M}_{\alpha,\beta}$ to be the *maximal* dihedral reflection subgroup whose associated root system $\Phi_{\mathcal{M}_{\alpha,\beta}}$ contains α and β. The simple roots of $\Phi_{\mathcal{M}_{\alpha,\beta}}$ is denoted $\Delta_{\mathcal{M}_{\alpha,\beta}}$.

2.1.8. *Short inversion posets.* We now recall an order on the short inversion roots introduced in [9]. Let $w \in W$. For $\alpha, \beta \in \Phi^1(w)$ define $\alpha \dot{\prec} \beta$ if $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$. This is equivalent to $\alpha \in \Delta_{\mathcal{M}_{\alpha,\beta}}$ and $\beta \notin \Delta_{\mathcal{M}_{\alpha,\beta}}$ (See [9, Proposition 3.2]). Then define the relation $\preceq_w$ as the transitive and reflexive closure of $\alpha \dot{\prec} \beta$. This is a partial order on $\Phi^1(w)$. The primary results we utilise here are the following.

Proposition 2.8. [9, Proposition 3.13] *The relation $\preceq_w$ is a partial order on $\Phi^1(w)$. Moreover, for any reduced word $w = s_1 s_2 \dots s_k$ consider the following total order $\leq$ on $\Phi(w)$:*

$$\alpha_{s_1} < s_1\alpha_{s_2} < \ldots < s_1 \ldots s_{k-1}\alpha_{s_k}$$

Then $\alpha \preceq_w \beta$ implies $\alpha \leq \beta$ and $dp(\alpha) \leq dp(\beta)$ for any $\alpha, \beta \in \Phi^1(w)$.

Theorem 2.6. [9, Theorem 3.17] *Let $w \in W$. For the poset $(\Phi^1(w), \preceq_w)$, the minimal elements are the left-descent roots in $\Phi^L(w)$ and the maximal elements are the right-descent roots in $\Phi^R(w)$. In other words, for any $\beta \in \Phi^1(w)$ there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \preceq_w \beta \preceq_w \gamma$.*

2.1.9. *The Cayley graph of (W, S).* We conclude our preliminary section with a brief review of the Cayley graph (which is only required in the proof of Theorem 2 and Proposition 3.2).

We denote X^1 to be the Cayley graph of W with vertex set $X^0 = W$ and edge-set $\{(w, ws) \mid w \in W, s \in S\}$ where each edge is of length 1. For vertices $x, y \in X^1$ a *geodesic* between x and y is a minimal length path between x and y in X^1. In the setting of X^1, for $w \in W$, $\ell(w)$ is the length of geodesics from the identity e to w.

It is well known that the left action of W on X^0 induces an action of W on X^1. For $r \in R$ the *wall* H_r is the fixed point set of r in X^1 and an edge (w, ws) *crosses* a wall H_r if and only if $r = wsw^{-1}$. Each wall H_r separates X^1 into two *half-spaces*. We denote H_r^+ to be the half space containing the identity and H_r^- the half space not containing the identity. By the bijective correspondence between roots and reflections, as subsets of W, we have $H_\beta^- = H_{s_\beta}^-$ (and $H_\beta^+ = H_{s_\beta}^+$). We will use the terms *walls* and *roots* interchangably.

Two distinct walls H_r and H_q *intersect* if H_r is not contained in a half-space for H_q (this relation is symmetric), or equivalently, $\langle r, q\rangle$ is a finite group. We say r, q are *sharp-angled* if r and q do not commute and there is $w \in W$ such that wrw^{-1} and wqw^{-1} are in S. Then there is a component of $X^1 \setminus (H_r \cup H_q)$ whose intersection with X^0 is a *geometric fundamental domain* for the action of $\langle r, q\rangle$ on X^0.

For $J \subseteq R$ and $p \in X^0$ the $J(p)$*-residue* is the subgraph of X^1 induced by the action of the subgroup generated by J on p. In particular, note that for $w \in W$ if $\alpha, \beta \in \Phi^R(w)$ then $w^{-1}s_\alpha w = s$ and $w^{-1}s_\beta w = t$ are in S and $\langle s_\alpha, s_\beta\rangle$ is a finite dihedral reflection subgroup with $2m(s, t)$ elements. Then the residue $\langle s_\alpha, s_\beta\rangle(w) = wW_{\langle s,t\rangle}$ and $w\Phi^+_{\langle s,t\rangle}$ is the set of $m(s, t)$ roots separating the $2m(s, t)$ elements of $\langle s_\alpha, s_\beta\rangle(w)$.

For the purposes of proving Theorem 2 and Proposition 3.2, we consider an order on paths in X^1 using the lexicographic order on tuples with entries in $\mathbb{N}$. For tuples $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, a_n)$ we have $\mathbf{a} \leq \mathbf{b}$ if and only if one of the following holds:

(i) There exists $j \leq \min(m, n)$ such that $a_i = b_i$ for all $i < j$ and $a_j < b_j$, or;
(ii) $a_i = b_i$ for all $i \leq m$ and $m < n$.

For a path $\pi = (p_0, \dots, p_n)$ in X^1 define the ordered tuple $\Pi = (n_L, \dots, n_1)$ where n_j is the number of elements $p_i \in \pi$ with $\ell(p_i) = j$ and $L = max_{i=0}^n \ell(p_i)$. Then for paths π, π' we have $\pi \leq \pi'$ if and only if $\Pi \leq \Pi'$.

3. Canonical join-representations

This section is dedicated to proving Theorem 1. We first give a few preliminary results. The following result is an obvious, but rather useful fact.

Lemma 3.1. *Let $x \in W$. If $\alpha \in \Phi(x)$ then there is $y \preceq x$ with $\Phi^R(y) = \{\alpha\}$*

Proof. Let x_0 be a prefix of x with $\alpha \in \Phi^R(x_0)$. If there is $\beta \in \Phi^R(x_0)$ with $\beta \neq \alpha$ then consider the prefix $x_1 := s_\beta x_0$ of x. Repeating this process, there must be some prefix x_k of x with $\Phi^R(x_k) = \{\alpha\}$. □

Proposition 3.1. *Let $x \in W$. Then*

$$x = \bigvee \phi^1(x)$$

where $\phi^1(x) = \{z \in W \mid z \preceq x$ with $\Phi^R(z) = \{\alpha\}$ and $\alpha \in \Phi^1(x)\}$.

Proof. Clearly x is an upper bound of $\phi^1(x)$. So by [2, Theorem 3.2.1] the join of $\phi^1(x)$ exists. Let $y = \bigvee \phi^1(x)$. Then $y \preceq x$ and by Proposition 2.3

$$\Phi(y) = cone_\Phi\big(\bigcup_{z\in\phi^1(x)} \Phi(z)\big) = cone_\Phi\big(\bigcup_{z\in\phi^1(x)} cone_\Phi \Phi^1(z)\big) = cone_\Phi\big(\bigcup_{z\in\phi^1(x)} \Phi^1(z)\big)$$

By Proposition 2.1 we have $\Phi^1(x) \subseteq \bigcup_{z\in\phi^1(x)} \Phi^1(z)$ and hence

$$\Phi(x) = cone_\Phi\big(\Phi^1(x)\big) \subseteq cone_\Phi\big(\bigcup_{z\in X} \Phi^1(z)\big) = \Phi(y)$$

Therefore $x \preceq y$. □

We note that nothing in Proposition 3.1 suggests that for each $\alpha \in \Phi^1(x)$ there cannot be more than one prefix z of x with $\Phi^R(z) = \{\alpha\}$. The following proposition shows that there is a bijection between the set $\phi^1(x) = \{z \in W \mid z \preceq x$ with $\Phi^R(z) = \{\alpha\}$ and $\alpha \in \Phi^1(x)\}$ and $\Phi^1(x)$.

Proposition 3.2. *Let $x \in W$. For each $\beta \in \Phi^1(x)$ there is a unique minimal length $z \preceq x$ such that $\Phi^R(z) = \{\beta\}$. Thus the map $\Phi_R : \phi^1(x) \to \Phi^1(x)$ defined by $z \mapsto \Phi^R(z)$ is a bijection.*

Proof. Let $\beta \in \Phi^1(x)$ and denote $Z_\beta = \{z \in W \mid z \preceq x$ and $\beta \in \Phi(z)\}$. We show that for each $z_0, z_n \in Z_\beta$ there is $z \in Z_\beta$ such that $z_0 \succeq z \preceq z_n$. Let $\pi = (z_0, z_1, \dots, z_n)$ be a geodesic edge path between z_0 and z_n. Note that

$$Z_\beta = \big(\bigcap_{\Phi^+\setminus\Phi(x)} H_\alpha^+\big) \cap H_\beta^-$$

and hence by Lemma 2.1 Z_β is convex. Therefore, each $z_i \in \pi$ is also in Z_β. Thus H_β does not separate any z_i from z_j in π and each z_i is a prefix of x.

Our aim is to modify π to obtain another embedded edge path between z_0 and z_n so that there is no $z_{i-1} \prec z_i \succ z_{i+1}$. Now suppose there is $z_{i-1} \prec z_i \succ z_{i+1}$. Let H_r, H_q be the walls separating z_{i-1} from z_i and z_i from z_{i+1} respectively and let α_r, α_q be the corresponding roots. Now z_{i-1}, z_{i+1} are elements of the *residue* $R = \langle r, q\rangle(z_i)$ and $\alpha_r, \alpha_q \in \Phi^R(z_i)$. Hence by [23, Theorem 2.9] we have $y \preceq z_i$ for all $y \in R$ and as a consequence we have that $y \preceq x$ for all $y \in R$. If r and q do not commute then r, q are sharp angled and the identity e is in a geometric fundamental domain for the finite dihedral reflection subgroup $W_{\langle r,q\rangle}$ generated by the reflections r, q. We claim that all the elements of the residue $R = \langle r, q\rangle(z_i)$ lie in Z_β.

We need to show that $\beta \notin \Phi^+_{\langle r,q\rangle}$. If r and q commute then we are done since there is no other root in $\Phi^+_{\langle r,q\rangle}$, otherwise if $\beta \in \Phi^+_{\langle r,q\rangle}$ then since $\beta = k\alpha_r + j\alpha_q$ for some $k, j \in \mathbb{R}_{>0}$ and $\alpha_r, \alpha_q \in \Phi(z_i) \subseteq \Phi(x)$ this implies $\beta \notin \Phi^1(x)$, a contradiction.

We now modify π and replace the subpath (z_{i-1}, z_i, z_{i+1}) with the other embedded edge-path in the residue R from z_{i-1} to z_{i+1}. Thus this path modification decreases the complexity of π (under lexicographic order); considering π as the tuple $\Pi = (n_L, \dots, n_2, n_1)$, where n_j is the number of elements z_i in π with $\ell(z_i) = j$ and $L = max_{i=0}^{n} \ell(z_i)$.

After finitely many such modifications we obtain the desired path. The fact that the minimal element z necessarily has $\Phi^R(z) = \{\beta\}$ follows by Lemma 3.1. □

The next result shows that the set $\phi^1(x)$ *join-refines* every join-representation Y of x.

Corollary 3.1. *Let $x \in W$ and suppose $x = \bigvee Y$ for $Y \subseteq W$. Then for every $z \in \phi^1(x)$ there is $y \in Y$ such that $z \preceq y$.*

Proof. Since $x = \bigvee Y$, by [16, Proposition 2.8 (4)] we have

$$\begin{aligned}\Phi(x) &= cone_\Phi\big(\bigcup_{y\in Y} \Phi(y)\big)\\ &= cone_\Phi\big(\bigcup_{y\in Y} cone_\Phi(\Phi^1(y))\big)\\ &= cone_\Phi\big(\bigcup_{y\in Y} \Phi^1(y)\big)\end{aligned}$$

Again by Proposition 2.1 we have $\Phi^1(x) \subseteq \bigcup_{y\in Y} \Phi^1(y)$ and thus for $\alpha \in \Phi^1(x)$, we have $\alpha \in \Phi^1(y)$ for some $y \in Y$. Hence by Proposition 3.2 there is $z \preceq y \preceq x$ with $\Phi^R(z) = \{\alpha\}$. Then again by Proposition 3.2, z is the unique element in $\phi^1(x)$ such that $\phi^R(z) = \{\alpha\}$ and the result follows. □

We can now prove Theorem 1.

Theorem 3.1. *Let $w \in W$. For each $\beta \in \Phi^R(w)$ there is a unique minimal length element $j_\beta \in \{v \mid v \preceq w, \beta \in \Phi(v)\}$. The canonical join representation of w is $w = \bigvee\{j_\beta \mid \beta \in \Phi^R(w)\}$.*

Proof. The first part of the theorem follows by Proposition 3.2 since $\Phi^R(w) \subseteq \Phi^1(w)$. For the second part, we first show that $w = \bigvee\{j_\beta \mid \beta \in \Phi^R(w)\}$. It suffices to show that for each $\alpha \in \Phi^1(w) \setminus \Phi^R(w)$ there is $\beta \in \Phi^R(w)$ such that $j_\alpha \preceq j_\beta$. Let $\alpha \in \Phi^1(w) \setminus \Phi^R(w)$. By Theorem 2.6 there is $\beta \in \Phi^R(w)$ such that $\alpha \preceq_w \beta$ and by Proposition 2.8 this implies that for any reduced word $\bar{w}$ of w we have $\alpha \leq \beta$ in the total order on $\Phi(w)$.

Now let $\bar{w}$ be a reduced word of w and let $\bar{w}_\beta$ be the initial path of $\bar{w}$ with $\beta \in \Phi^R(\bar{w}_\beta)$, where $\Phi^R(\bar{w}_\beta)$ is the right-descent set of the element represented by the word $\bar{w}_\beta$. Note that $\bar{w}_\beta$ is a reduced word since $\bar{w}$ is reduced. By Proposition 3.2 $j_\beta \preceq \bar{w}_\beta$ and so we can modify $\bar{w}$ to obtain another reduced word of w, $\bar{\bar{w}}$ with j_β the initial path of $\bar{\bar{w}}$ with $\beta \in \Phi^R(j_\beta)$. By Proposition 2.8 again, for the reduced word $\bar{\bar{w}}$ there must be an initial path of $\bar{\bar{w}}$ with reduced word $\bar{w}_\alpha$ with $\alpha \in \Phi^R(\bar{w}_\alpha)$ such that $w_\alpha \preceq j_\beta \preceq w$. By Proposition 3.2 again, we obtain $j_\alpha \preceq w_\alpha \preceq j_\beta \preceq w$ as required.

Let $\phi^R(w) = \{j_\beta \mid \beta \in \Phi^R(w)\}$. We now show that $\phi^R(w)$ is irredundant. If $|\Phi^R(w)| = 1$, then w is join-irreducible and the result is clear, hence assume $|\Phi^R(w)| > 1$. Suppose for some $\alpha \in \Phi^R(w)$ we have

$$\begin{aligned}w &= \bigvee\{j_\beta \mid \beta \in \Phi^R(w) \setminus \{\alpha\}\}\\ &= \bigvee\{j_\beta \mid \beta \in \Phi^1(w) \setminus \{\alpha\}\} \quad \text{by Proposition 3.1}\end{aligned}$$

It then follows that $\Phi^1(s_\alpha w) = \Phi^1(w) \setminus \{\alpha\}$ and thus

$$w \succ s_\alpha w = \bigvee\{j_\beta \mid \beta \in \Phi^1(w) \setminus \{\alpha\}\} \qquad \text{again by Proposition 3.1}$$

a contradiction. Finally, the fact that the set $\phi^R(w)$ join-refines every join representation of w follows by Corollary 3.1 since $\phi^R(w) \subseteq \phi^1(w)$. □

4. Witnesses of boundary roots

In this section, we show that for any element $x \in W$ and $\beta \in \Phi(x)$, the set of elements $y \in W$ satisfying $\Phi(y) \cap \Phi(x) = \{\beta\}$ is convex in the weak order and admits a unique minimal representative. By Lemma 1.1 these are the *witnesses* of β with respect to x and means that β is a *boundary root* of the cone type $T = T(x^{-1})$.

Furthermore, we show that this minimal witness is the same one for all $z \in W$ with $T(z^{-1}) = T$ (i.e. all $z \in Q(T)$). In addition, we show that for any Garside shadow G, the set G^0 is closed under suffix. In this section, for $w \in W$ let $\pi(w)$ be the Γ-projection of w. We begin with some basic properties of the set of witnesses.

Lemma 4.1. *Let $w \in W$ and $\beta \in \partial T(w^{-1})$. Then w is it's own witness of β if and only if $w \in S$.*

Proof. If w is it's own witness of β, then $\Phi(w) \cap \Phi(w) = \{\beta\}$. Clearly, this can only be true if $w \in S$. □

Lemma 4.2. *Let $x \in W$ and $T := T(x^{-1})$. Then*

$$\partial T(x^{-1})_\beta = H_\beta^- \cap \big(\bigcap_{\alpha \in \Phi(x) \setminus \{\beta\}} H_\alpha^+\big)$$

and the set $\partial T(x^{-1})_\beta$ is convex.

Proof. The proof is straightforward. If $w \in \partial T(x^{-1})_\beta$ then $\alpha \notin \Phi(w)$ for all $\alpha \neq \beta \in \Phi(x)$, so $w \in H_\alpha^+$ for all $\alpha \in \Phi(x) \setminus \{\beta\}$. Since $\beta \in \Phi(w)$ we have $w \in H_\beta^-$. The reverse inclusion is also clear. Convexity follows by Lemma 2.1 since $\partial T(x^{-1})_\beta$ is an intersection of half spaces. □

Lemma 4.3. *Let $T \in \mathbb{T}$, $\beta \in \partial T$ and $x \in Q(T)$. If $w \in \partial T(x^{-1})_\beta$ then $v \in \partial T(x^{-1})_\beta$ for all $\pi(w) \preceq v \preceq w$.*

Proof. Let $w \in \partial T(x^{-1})_\beta$. So $\Phi(x) \cap \Phi(w) = \{\beta\}$ and hence $\beta \in \partial T(w^{-1})$. Since $\pi(w) \preceq v \preceq w$ and $\beta \in \Phi(\pi(w))$ we have $\Phi(x) \cap \Phi(v) = \{\beta\}$ for all $\pi(w) \preceq v \preceq w$, so $v \in \partial T(x^{-1})_\beta$. □

The next result shows that the set of witnesses have a similar property to cone types in terms of prefix relation (see Lemma 2.8).

Lemma 4.4. *Let $T \in \mathbb{T}$ and $x, y \in Q(T)$. If $\beta \in \partial T$ and $x \preceq y$ then $\partial T(y^{-1})_\beta \subseteq \partial T(x^{-1})_\beta$.*

Proof. If $w \in \partial T(y^{-1})_\beta$ then $\Phi(y) \cap \Phi(w) = \{\beta\}$ and since $x \preceq y$ with $\beta \in \Phi(x)$ then $\Phi(x) \cap \Phi(w) = \{\beta\}$ and so $w \in \partial T(x^{-1})_\beta$. □

Corollary 4.1. *Let $T \in \mathbb{T}$ and x the minimal element of $Q(T)$. Then*

$$\partial T(x^{-1})_\beta = \bigcup_{y \in Q(T)} \partial T(y^{-1})_\beta$$

Proof. Since $x \in Q(T)$, the inclusion $\partial T(x^{-1})_\beta \subseteq \bigcup_{y \in Q(T)} \partial T(y^{-1})_\beta$ is clear. The reversion inclusion follows from Lemma 4.4 since $x \preceq y$ for all $y \in Q(T)$. □

The following two results are not directly required for our proof of Theorem 2. They are included to demonstrate how the boundary roots and witnesses can be inductively computed.

Lemma 4.5. [26, Lemma 3.3.4] *Let $w \in W$ and $s \in S$ where $s \notin D_L(w)$. Then*

$$\partial T(w^{-1}s) = \{\alpha_s\} \sqcup s\big(\{\alpha \in \partial T(w^{-1}) \mid \exists\, x \in \partial T(w^{-1})_\alpha \text{ with } s \in D_L(x)\}\big)$$

Proof. We first show the left to right inclusion. If $\alpha \neq \alpha_s \in \partial T(w^{-1}s)$ then by Proposition 2.5 (i) there is $x \in W$ such that $\Phi(sw) \cap \Phi(x) = \{\alpha\}$. Thus $\alpha_s \notin \Phi(x)$ and hence $\Phi(sx) = \{\alpha_s\} \sqcup s\Phi(x)$. Since $\alpha_s \in \Phi(sw)$ we have $\Phi(w) = s\big(\Phi(sw) \setminus \{\alpha_s\}\big)$. Therefore

$$\begin{aligned}\Phi(w) \cap \Phi(sx) &= s\big(\Phi(sw) \setminus \{\alpha_s\}\big) \cap \big(\{\alpha_s\} \sqcup s\Phi(x)\big)\\ &= s\big(\Phi(sw) \cap \Phi(x)\big)\\ &= s(\{\alpha\})\end{aligned}$$

We now show the right to left inclusion. It is clear that $\alpha_s \in \partial T(w^{-1}s)$. For a root $\alpha \in \partial T(w^{-1})$ where there is $x \in W$ with $\Phi(w) \cap \Phi(x) = \{\alpha\}$ and $s \in D_L(x)$ we aim to show that $\Phi(sw) \cap \Phi(sx) = \{s\alpha\}$. It then follows by Proposition 2.5 (ii) that $s\alpha$ is a boundary root of $T(w^{-1}s)$. Since $\ell(sw) > \ell(w)$ we have that $\Phi(sw) = \{\alpha_s\} \sqcup s\Phi(w)$. If $x \in W$ is such that $s \in D_L(x)$ then $\Phi(sx) = s\big(\Phi(x) \setminus \{\alpha_s\}\big)$. Hence we have

$$\Phi(sw) \cap \Phi(sx) = \big(\{\alpha_s\} \sqcup s\Phi(w)\big) \cap s\big(\Phi(x) \setminus \{\alpha_s\}\big)$$

Since $\Phi(w) \cap \Phi(x) = \{\alpha\}$ by the formula for $\Phi(sw) \cap \Phi(sx)$ above it follows that $s\alpha \in \Phi(sw) \cap \Phi(sx)$. We now show that $\Phi(sw) \cap \Phi(sx) = \{s\alpha\}$. If $\beta \in \Phi(sw) \cap \Phi(sx)$ then $\beta \in s\Phi(w) \cap s\Phi(x) = s\big(\Phi(w) \cap \Phi(x)\big) = s(\{\alpha\})$. So $\beta = s\alpha$ as required. □

The next result shows how the set $\partial T(sx^{-1})_{s\beta}$ is obtained from $\partial T(x^{-1})_\beta$ if $\beta \in \partial T(x^{-1})$ and $s\beta \in \partial T(sx^{-1})$.

Corollary 4.2. *Let $x \in W$, $s \in S$ with $s \notin D_L(x)$. Let $y = sx$. If $\beta \in \partial T(x^{-1})$ and $s\beta \in \partial T(y^{-1})$ then*

$$\partial T(y^{-1})_{s\beta} = \{w \in \partial T(x^{-1})_\beta \mid s \in D_L(w)\}$$

Proof. If $v \in \partial T(y^{-1})_{s\beta}$ then $\Phi(y) \cap \Phi(v) = \{s\beta\}$. Since $\alpha_s \in \Phi(y)$ this implies that $\alpha_s \notin \Phi(v)$ and so $\Phi(sv) = \{\alpha_s\} \sqcup s\Phi(v)$. Since $\Phi(x) = s\Phi(y) \setminus \{\alpha_s\}$ we have

$$\Phi(x) \cap \Phi(sv) = \big(s\Phi(y) \setminus \{\alpha_s\}\big) \cap \big(\{\alpha_s\} \sqcup s\Phi(v)\big)$$

Therefore, as in the proof of Lemma 4.5, we have $\Phi(x) \cap \Phi(sv) = \{\beta\}$. The reverse inclusion directly follows from the second half of the proof of Lemma 4.5. □

We now prove the main result of this section.

Theorem 4.1. *Let $x \in W$ and $\beta \in \Phi(x)$ such that the set $\{y' \in W \mid \Phi(x) \cap \Phi(y') = \{\beta\}\}$ is non-empty. Then there exists a unique minimal length element $y \in W$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

Furthermore, $y \in \Gamma^0$ and the set of witnesses $\partial T(x^{-1})_\beta$ is a convex, gated set in W.

Proof. The proof is very similar to the proof of Proposition 3.2. We pass between reflections in the Cayley graph X^1 of W and their corresponding roots in Φ^+. Hence consider $\partial T(x^{-1})_\beta$ to be the intersection of the half spaces H_t^- $(t := s_\beta)$ and $H_{u_i}^+$ $(u_i := s_{\alpha_i})$ in X^1 corresponding to the roots β and $\{\alpha_i \mid \alpha_i \in \Phi(x) \setminus \{\beta\}\}$ (per the characterisation of $\partial T(x^{-1})_\beta$ in Lemma 4.2).

It suffices to show that for each $p_0, p_n \in \partial T(x^{-1})_\beta$ there is $p \in \partial T(x^{-1})_\beta$ with $p_0 \succeq p \preceq p_n$. Let $\pi = (p_0, p_1, \ldots, p_n)$ be a geodesic edge path between p_0 and p_n. By Lemma 4.2, each $p_i \in \partial T(x^{-1})_\beta$ for $0 \leq i \leq n$.

Our aim is to modify π to obtain another embedded edge path between p_0 and p_n (not necessarily geodesic) so that there is no $p_{i-1} \prec p_i \succ p_{i+1}$. Now suppose there is $p_{i-1} \prec p_i \succ p_{i+1}$. Let H_r, H_q be the walls separating p_{i-1} from p_i and p_i from p_{i+1} respectively and let α_r, α_q be the corresponding roots. Note that $\alpha_r \neq \beta$ and $\alpha_q \neq \beta$ since by convexity $p_{i-1}, p_i, p_{i+1} \in \partial T(x^{-1})_\beta$. Also, p_{i-1}, p_{i+1} are elements of the *residue* $R = \langle r, q\rangle(p_i)$ and $\alpha_r, \alpha_q \in \Phi^R(p_i)$. If r and q do not commute then r, q are sharp angled and the identity e is in a geometric fundamental domain for the finite dihedral reflection subgroup $W_{\langle r,q\rangle}$ generated by the reflections r, q. We claim that all the elements of the residue $R = \langle r, q\rangle(p_i)$ lie in $\partial T(x^{-1})_\beta$.

Since $p_i \in \partial T(x^{-1})_\beta$ this implies that $\alpha_r, \alpha_q \notin \Phi(x) \setminus \{\beta\}$. We then need to show that $\alpha' \notin \Phi(x) \setminus \{\beta\}$ for any $\alpha' \in \Phi^+_{\langle r,q\rangle}$. Hence we can assume that r and q do not commute, otherwise there is no other root in $\Phi^+_{\langle r,q\rangle}$. If there is $\alpha' \in \Phi^+_{\langle r,q\rangle} \cap \big(\Phi(x) \setminus \{\beta\}\big)$ then since $\alpha' = k\alpha_r + j\alpha_q$ for some $k, j \in \mathbb{R}_{>0}$ this implies that either $\alpha_r \in \Phi(x) \cap \Phi(p_i)$ or $\alpha_q \in \Phi(x) \cap \Phi(p_i)$. Since $\beta \in \Phi(x) \cap \Phi(p_i)$, this then implies that $|\Phi(x) \cap \Phi(p_i)| > 1$, a contradiction.

We now modify π and replace the subpath (p_{i-1}, p_i, p_{i+1}) with the other embedded edge-path in the residue R from p_{i-1} to p_{i+1}. By [23, Theorem 2.9] all the elements of R are $\preceq p_i$, since the element $\mathrm{Proj}_R(e)$ of R is opposite to p_i. It then follows by [23, Theorem 2.15] that all elements of R lie on a geodesic edge path from $\mathrm{Proj}_R(e)$ to p_i and hence on a geodesic edge path from e to p_i through $\mathrm{Proj}_R(e)$. Thus this path modification decreases the complexity of π (under lexicographic order); considering π as the tuple $\Pi = (n_L, \ldots, n_2, n_1)$, where n_j is the number of elements p_i in π with $\ell(p_i) = j$ and $L = max_{i=0}^n \ell(p_i)$.

After finitely many such modifications we obtain the desired path. The fact that the minimal element y is necessarily a tight gate follows by Proposition 2.6 and convexity follows by Lemma 4.2. □

Corollary 4.3. *For each $T \in \mathbb{T}$ and $\beta \in \partial T$, there is a unique minimal length element $y \in W$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

for all $x \in Q(T)$

Proof. Let $u, v \in Q(T)$ and $\beta \in \partial T$. By Theorem 4.1 there are unique minimal length elements u', v' such that $\Phi(u) \cap \Phi(u') = \{\beta\}$ and $\Phi(v) \cap \Phi(v') = \{\beta\}$. Let g be the minimal element of $Q(T)$. Then $g \preceq u$ and $g \preceq v$ and $\Phi(g) \cap \Phi(u') = \{\beta\}$ and $\Phi(g) \cap \Phi(v') = \{\beta\}$. If $u' \neq v'$ then applying Theorem 4.1 to g, there is $z \in W$ with $z \prec u'$ and $z \prec v'$ such that $\Phi(g) \cap \Phi(z) = \{\beta\}$. But then also $\Phi(u) \cap \Phi(z) = \{\beta\}$ and $\Phi(v) \cap \Phi(z) = \{\beta\}$ contradicting he minimality of u' and v'. Hence we must have $u' = v'$. □

As a consequence of Theorem 4.1 we also obtain the following properties for ∂T_β.

Corollary 4.4. *For each $T \in \mathbb{T}$, the following properties hold:*

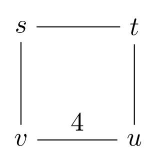
(i) $\partial T_\beta \neq \emptyset$.
(ii) $\partial T_\beta = \bigcap_{y \in Q(T)} \partial T(y^{-1})_\beta$.
(iii) ∂T_β *is convex and gated.*

Proof. (i) This directly follows from Corollary 4.3.
(ii) Follows by definition of ∂T_β.
(iii) This follows by (i) and (ii) and by the fact that each $\partial T(y^{-1})_\beta$ is convex.

□

The results in this section show that for any cone type T and $\beta \in \partial T$, there is a unique tight gate y that is a witness to β for all $x' \in Q(T)$. By symmetry, there is a unique tight gate x with $\Phi(y) \cap \Phi(x) = \{\beta\}$ and hence there is a natural "pairing" of tight gates with super elementary roots. The following example demonstrates that there are potentially multiple pairs of minimal-length tight gates x, y with $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^R(x) = \Phi^R(y) = \{\beta\}$ for a super-elementary root β.

Example 4.6. Let W be the rank 4 compact hyperbolic Coxeter group whose graph is given by



Let $\beta = \alpha_t + \sqrt{2}\alpha_u + 2\alpha_v$. Denote the elements $a = tvutv, b = uvutv, c = utvutv, d = vutv$. Then these elements are all tight gates with $\Phi^R(a) = \Phi^R(b) = \Phi^R(c) = \Phi^R(d) = \{\beta\}$ and we have

$$\Phi(a) \cap \Phi(b) = \{\beta\}$$
$$\Phi(c) \cap \Phi(d) = \{\beta\}$$

But d is not a witness of β with respect to a and b is not a witness of β with respect to c.

We conclude this section by showing that the *tight elements* of a Garside shadow G are closed under suffix. Note that we loosely use the term "tight elements" here rather than *tight gates* since we are not considering the partition induced by G.

Lemma 4.7. *Let $x \in W$ with $\ell(x) > 1$ and $|\Phi^R(x)| = 1$. If $s \in D_L(x)$ then $\Phi^R(sx) = \{-sx\alpha_s \mid s \in D_R(x)\}$ and $|\Phi^R(sx)| = 1$.*

Proof. By Lemma 2.3, it suffices to show that $D_R(sx) = D_R(x)$. Since $|\Phi^R(x)| = |D_R(x)| = 1$ there is a single simple reflection $t \in D_R(x)$. Since $\ell(x) > 1$, any reduced expression for x must be written as $x = s_1 \dots s_k t$ where $k \geq 1$. Hence, if $s \in D_L(x)$ then any reduced expression for sx must end in t. Thus, $t \in D_R(sx)$. The other inclusion is clear. □

Theorem 4.2. *Let G be a Garside shadow and*

$$G^0 = \{x \in G \mid |\Phi^R(x)| = 1\}$$

Then the set G^0 is closed under suffix.

Proof. Let $x \in G^0$, then since G is closed under suffix, if $s \in D_L(x)$ then $sx \in G$ and by Lemma 4.7 $sx \in G^0$. □

Corollary 4.5. *The sets L_m^0 and Γ^0 are closed under suffix.*

Proof. This follows immediately from Theorem 4.2. □

The following result also implies that the set of tight gates Γ^0 is closed under suffix and gives more precise information about the relationship between tight gates and super-elementary roots.

Proposition 4.1. *Let $x \in \Gamma$ and $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^R(y) = \{\beta\}$ where y is of minimal length with this property. If $s \in D_L(y)$ then*

$$\Phi(sx) \cap \Phi(sy) = \{s\beta\}$$

and $sy \in \Gamma^0$.

Proof. Let $x, y \in W$ possess the properties as stated. Since $\alpha_s \in \Phi(y)$ it follows that $\alpha_s \notin \Phi(x)$. Denote $v = sx$. It then follows that $\Phi(v) = \{\alpha_s\} \sqcup s\Phi(x)$ and $\Phi(sy) = s\Phi(y) \setminus \{\alpha_s\}$. Therefore $\Phi(v) \cap \Phi(sy) = \{s\beta\}$ and it is clear that $s\beta \in \Phi^R(sy)$. Now suppose there is $\alpha \neq s\beta \in \Phi^R(sy)$. Then we have $s_\alpha sy \preceq sy$ and $\alpha_s \notin \Phi(s_\alpha sy) \cap \Phi(sy)$. Thus $ss_\alpha sy \preceq y$ with $\ell(s_\alpha sy) = \ell(sy) - 1 = \ell(y) - 2$ and $s\beta \in \Phi(s_\alpha sy)$. Then $\Phi(ss_\alpha sy) = \{\alpha_s\} \sqcup \Phi(s_\alpha sy)$ and $\beta \in \Phi(ss_\alpha sy)$. This then implies that

$$\Phi(x) \cap \Phi(ss_\alpha sy) = \{\beta\}$$

and $\ell(ss_\alpha sy) = \ell(y) - 1$ contradicting the minimality of y. □

5. The $\mathscr{T}^0$ partition and join-irreducible elements of L_m and Γ

In this section, we initiate the study of some properties of the tight gates Γ^0 of $\mathscr{T}$. We show that the partition induced by the tight gates Γ^0 produces the cone type arrangement $\mathscr{T}$. We also show that the sets L_m^0 and Γ^0 are the join-irreducible elements of the posets L_m and Γ under the right weak order. In particular, in follows from [22, Theorem 1.4] that the set Γ^0 are the most fundamental elements of the smallest Garside shadow of W.

Lemma 5.1. *Let $x \in W$. Then the set of prefixes of x, $P(x) = \{z \preceq x \mid z \in W\}$ is convex.*

Proof. Clearly, we have $P(x) = \bigcap_{\Phi^+ \setminus \Phi(x)} H_\alpha^+$. Hence the result follows by Lemma 2.1. □

The following result shows that every gate of $\mathscr{T}$ is a join of tight gates.

Theorem 5.1. *Let $x \in \Gamma$. Then*

$$x = \bigvee \partial t(x^{-1})$$

where $\partial t(x^{-1}) = \{z \in W \mid z \preceq x \text{ with } \Phi^R(z) = \{\alpha\} \text{ and } \alpha \in \partial T(x^{-1})\}$. Hence, every gate is a join of tight gates.

Proof. Let $x \in \Gamma$. By Lemma 3.1, for each $\alpha \in \partial T(x^{-1})$, there is $z_\alpha \preceq x$ with $\Phi^R(z_\alpha) = \{\alpha\}$. Since $\alpha \in \partial T(x^{-1})$, there exists $w_\alpha \in \Gamma^0$ such that $\Phi(x) \cap \Phi(w_\alpha) = \{\alpha\}$. Therefore, we also have $\Phi(z_\alpha) \cap \Phi(w_\alpha) = \{\alpha\}$. By Theorem 2.3 then we have $z_\alpha \in \Gamma^0$ and x is an upper bound of

$$\partial t(x^{-1}) := \{z \in W \mid z \preceq x \text{ with } \Phi^R(z) = \{\alpha\} \text{ and } \alpha \in \partial T(x^{-1})\} \subset \Gamma^0$$

Now let $y = \bigvee \partial t(x^{-1})$ and let $\Phi(\partial t(x^{-1})) := \bigcup_{z \in \partial t(x^{-1})} \Phi(z)$. By Proposition 2.4 and Theorem 2.5 we have

$$T(y^{-1}) = \bigcap_{z \in \partial t(x^{-1})} T(z^{-1}) = \bigcap_{\Phi(\partial t(x^{-1}))} H_\beta^+$$

Since $y \preceq x$ and $\partial T(x^{-1}) \subseteq \Phi(\partial t(x^{-1}))$ it follows by Theorem 2.5 again that

$$T(x^{-1}) \subseteq T(y^{-1}) = \bigcap_{\Phi(\partial t(x^{-1}))} H_\beta^+ \subseteq \bigcap_{\partial T(x^{-1})} H_\beta^+ = T(x^{-1})$$

So $T(x^{-1}) = T(y^{-1})$ and since $x \in \Gamma$, by Theorem 2.2 we must have $y = x$. □

Corollary 5.1. *For $x \in W$, $\partial t(x^{-1}) \subseteq \phi^1(x)$.*

Proof. Proposition 3.2 shows that for each $\beta \in \Phi^1(x)$ there is a unique minimal length element $j_\beta \in \phi^1(x)$. Since $\partial T(x^{-1}) \subseteq \Phi^1(x)$ the result follows by Theorem 5.1 and Proposition 3.1. □

Corollary 5.2. *Each cone type $T \in \mathbb{T}$ is expressible as an intersection of cone types of tight gates.*

Proof. By Theorem 5.1 it follows that every gate is a join of a set of tight gates $\partial t(x^{-1})$. Then by Proposition 2.4 we have that $T(x^{-1}) = \bigcap_{z \in \partial t(x^{-1})} T(z^{-1})$. □

Corollary 5.3. *Define $x \sim y$ if the following holds: $x \in T(w^{-1})$ if and only if $y \in T(w^{-1})$ for all $w \in \Gamma^0$.*

Let $\mathscr{T}^0$ be the partition of W with parts P defined by the relation $x \sim y$. Then $\mathscr{T}^0 = \mathscr{T}$ and hence $\mathscr{T}^0$ is a regular partition.

Proof. By Theorem 2.1 it suffices to show that if x, y are in the same part of $\mathscr{T}^0$ then x, y are in the same part of $\mathscr{T}$. Suppose $x \in T(z^{-1})$ if and only if $y \in T(z^{-1})$ for all $z \in \mathscr{T}^0$. Let $x \in T(g^{-1})$ for $g \in \Gamma \setminus \Gamma^0$. By Theorem 5.1 and Proposition 2.4 we then have $x \in T(w^{-1})$ for all $w \in \partial t(g^{-1})$ where

$$\partial t(g^{-1}) = \{w \in W \mid w \preceq g \text{ with } \Phi^R(w) = \{\alpha\} \text{ and } \alpha \in \partial T(g^{-1})\}$$

Since $\partial t(g^{-1}) \subset \Gamma^0$ this implies $y \in T(w^{-1})$ for all $w \in \partial t(g^{-1})$ and hence by Theorem 5.1 and Proposition 2.4 again this implies $y \in T(g^{-1})$. Similarly, if $x \notin T(g^{-1})$ for some $g \in \Gamma \setminus \Gamma^0$ then by Theorem 5.1, there must be some $w \preceq g$ with $\Phi^R(w) = \{\alpha\}$ and $\alpha \in \partial T(g^{-1}) \cap \partial T(w^{-1})$ such that $x \notin T(w^{-1})$. Therefore, also $y \notin T(w^{-1})$. □

Remark 5.1. A consequence of Corollary 5.3 is that to determine whether two elements x, y have the same cone type (or equivalently, whether x^{-1} and y^{-1} live in the same part of the cone type partition), one only needs to compute the set of tight gates Γ^0 and then check whether $x \in T(w^{-1})$ if and only if $y \in T(w^{-1})$ for all $w \in \Gamma^0$. Our computations using Sagemath ([7]) in Section 6.1 show that often, the set Γ^0 is much smaller than Γ. We illustrate the $\mathscr{T}^0$ arrangements for the rank 3 Coxeter groups of affine type in Figure 4.

Corollary 5.4. *Every $x \in L_m$ is a join of tight gates of $\mathscr{S}_m$.*

Proof. By definition, if $x \in L_m$ then $\Phi^1(x) \subseteq \mathscr{E}_m$. Hence, for each $z \in \phi^1(x)$ as defined in Proposition 3.1 we have $\Phi^R(z) \subseteq \mathscr{E}_m$. Thus it follows by [9, Theorem 4.8] that $\phi^1(x) \subseteq L_m^0$. □

Recall that for a poset P an element $x \in P$ is *join-irreducible* if x is not the minimal element in P and if $x = a \vee b$ then either $x = a$ or $x = b$. Equivalently, if $X \subseteq P$ and $x = \bigvee X$ then $x \in X$.

Theorem 5.2. *The set*

$$L_m^0 = \{x \in L_m \mid |\Phi^R(x)| = 1\}$$

is the set of join-irreducible elements of $(L_m, \preceq)$.

Proof. Let $x \in L_m$ with $\Phi^R(x) = \{\beta\}$. If x is not join-irreducible, then $x = \bigvee A$ for some $A \subseteq L_m$ with $x \notin A$. Since $\beta \in \Phi^1(x)$ by Corollary 2.1 $\beta \in \Phi^1(y)$ for some $y \in A$ and $y \prec x$. Therefore, there must be $\alpha \neq \beta \in \Phi^R(x)$, a contradiction.

Now suppose x is join-irreducible in L_m and $|\Phi^R(x)| > 1$. Let $\{\alpha_1, \dots, \alpha_k\} = \Phi^R(x)$ and $x_i := s_{\alpha_i} x$ and let $y_i := \pi(x_i) \preceq x_i$ be the L_m projection of x_i (i.e. y_i is the minimal element of the part containing x_i in the m-Shi partition). We claim that $x = \bigvee y_i$. Note that since $y_i \in L_m$ we have

$$\Phi(y_i) = cone_\Phi(\mathscr{E}_m(y_i))$$

Since each $\alpha_i \in \mathscr{E}_m$ we have

$$\mathscr{E}_m(x_i) = \mathscr{E}_m(y_i) = \mathscr{E}_m(x) \setminus \{\alpha_i\}$$

and $\mathscr{E}_m(x) = \bigcup \mathscr{E}_m(y_i)$. Then since $x \in L_m$ we have

$$\Phi(x) = cone_\Phi(\mathscr{E}_m(x)) = cone_\Phi\big(\bigcup \mathscr{E}_m(y_i)\big)$$

Now let $y = \bigvee y_i$. Then by Proposition 2.3

$$\begin{aligned}\Phi(y) &= cone_\Phi\big(\bigcup \Phi(y_i)\big)\\ &= cone_\Phi\big(\bigcup cone_\Phi(\mathscr{E}_m(y_i))\big)\\ &= cone_\Phi\big(\bigcup \mathscr{E}_m(y_i)\big)\\ &= \Phi(x)\end{aligned}$$

where the equality of the second and third lines above follows from [16, Proposition 2.8 (4)]. Hence $x = y$. □

A similar argument to Theorem 5.2 follows for the set of tight gates Γ^0.

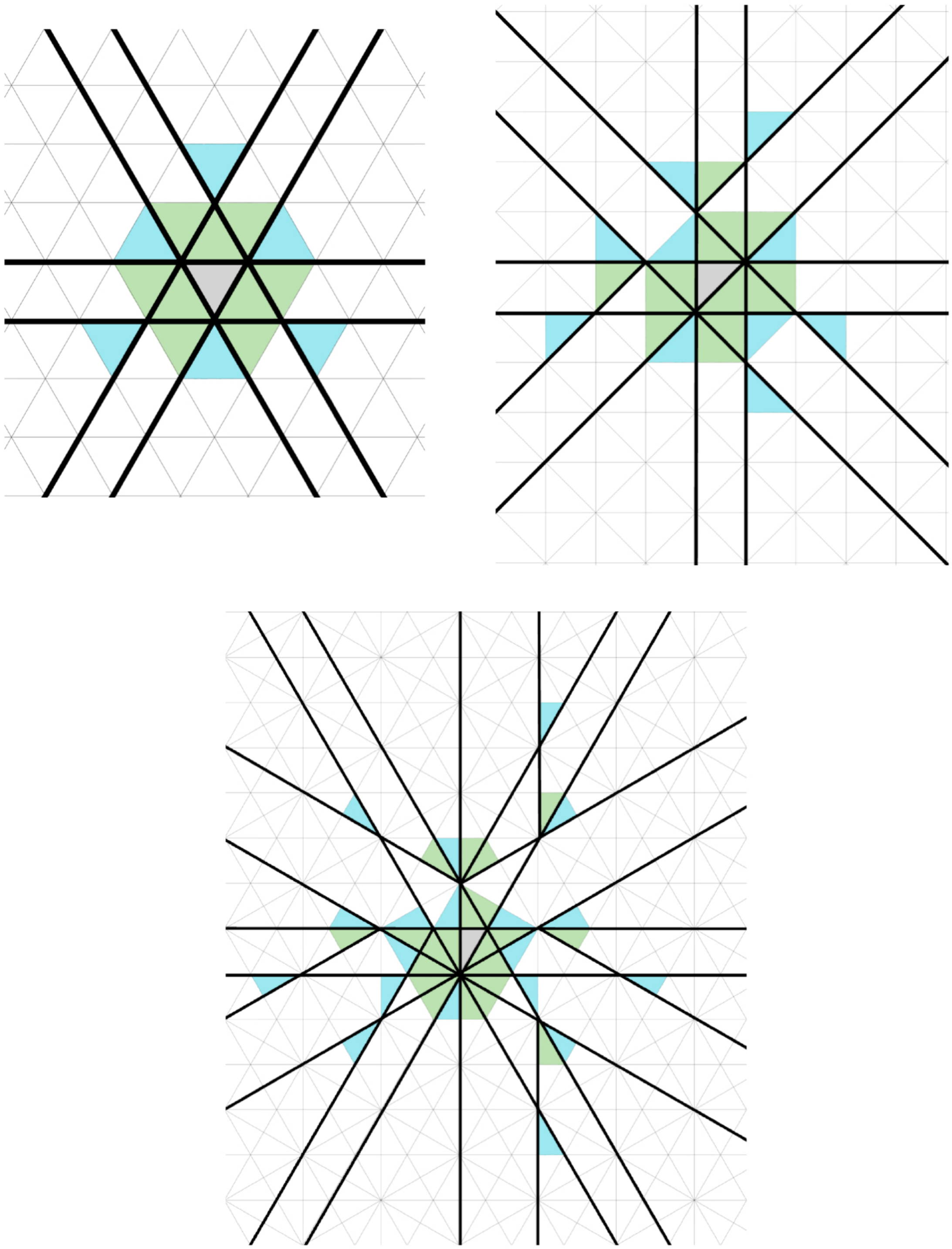


FIGURE 4. The partition $\mathscr{T}^0 = \mathscr{T}$ for the rank 3 Coxeter groups of affine type. The green alcoves represent the tight gates and the cyan alcoves are the gates which are non-trivial joins of tight gates.

Theorem 5.3. *The set*

$$\Gamma^0 = \{x \in \Gamma \mid |\Phi^R(x)| = 1\}$$

is the set of join-irreducible elements of $(\Gamma, \preceq)$.

Proof. When $x \in \Gamma^0$ and $\Phi^R(x) = \{\beta\}$ the argument is exactly the same as the first part of Theorem 5.2.

Now let $x \in \Gamma$ be join-irreducible in $(\Gamma, \preceq)$ and suppose $|\Phi^R(x)| > 1$. Let $\{\alpha_1, \dots \alpha_k\} = \Phi^R(x)$ and $x_i := s_{\alpha_i} x$. Again let $\pi(x_i) \in \Gamma$ be the Γ-projection of x_i (i.e. $\pi(x_i)$ is the minimal element of the part containing x_i in the cone type partition $\mathscr{T}$). We claim that $x = \bigvee \pi(x_i)$. Let $y = \bigvee \pi(x_i)$. Note that since x is an upper bound for each $\pi(x_i)$, this implies $y \preceq x$. Then since $x = \bigvee x_i$ and $T(x_i^{-1}) = T(\pi(x_i)^{-1})$ for each i, by Proposition 2.4 we have

$$T(x^{-1}) = \bigcap_{x_i} T(x_i^{-1}) = \bigcap_{x_i} T(\pi(x_i)^{-1}) = T(y^{-1})$$

Since $x \in \Gamma$ we must have $x = y$. □

6. Computing the sets $\mathcal{S}$ and Γ^0

Super elementary roots and tight gates are intimately connected by definition. In this section, we highlight some results connecting the relationship between the two.

Since $\mathcal{S} \subseteq \mathscr{E}$, the set $\mathcal{S}$ can be determined in finite time by considering each root $\beta \in \mathscr{E}$ and checking whether there exists a pair of gates or low-elements $\{x, y\}$ with $\Phi(x) \cap \Phi(y) = \{\beta\}$. An application of Theorem 4.1 and Corollary 4.5 gives an alternative method which concurrently computes the set $\mathcal{S}$ and Γ^0 without first having to compute the set L or the set Γ. It only requires that the elementary roots $\mathscr{E}$ is computed first (which is quite efficient).

Algorithm 6.1 Compute tight gates and super-elementary roots

Let $L' = S$, $\mathcal{S}' = \Delta$, $\Gamma' = \emptyset$
Initialize $L'_i = \{x \in L' \mid \ell(x) = i\}$ with $i = 1$
while $L'_i \neq \emptyset$ **do**
  for $x \in L'_i$ **do**
    for $s \notin D_L(x)$ **do**
      Set $y = s \cdot x$
      if $|\Phi^R(y)| = 1$ and $\{\beta\} = \Phi^R(y) \in \mathcal{E}$ **then**
        Add y to L'
        if there is $z \neq y \in L'$ with $\Phi^R(z) = \{\beta\}$ and $\Phi(z) \cap \Phi(y) = \{\beta\}$ **then**
          Add β to $\mathcal{S}'$
          Add y, z to Γ'
        end if
      end if
    end for
  end for
  Set $i = i + 1$
end while

The algorithm terminates in finite time and upon termination $\Gamma' = \Gamma^0$ and $\mathcal{S}' = \mathcal{S}$

Proof. By Corollary 4.5 it follows that the tight gates form saturated chains in the partial order $(\Gamma, \preceq)$. The set L' in the algorithm iteratively records the set $L^0 = \{w \in L \mid |\Phi^R(w)| = 1\}$. Since $\Gamma^0 \subseteq L^0 \subseteq L$ and $|L| < \infty$, the algorithm terminates in finite time. Inductively, if $x \in \Gamma'$, then $x \in \Gamma^0$ and if $y = s \cdot x \in \Gamma^0$ then there is some $z \in L'$ such that $\Phi(y) \cap \Phi(z) = \{\beta\}$ with $\Phi^R(y) = \{\beta\} = \Phi^R(z)$. So $\beta \in \mathcal{S}$ and by Theorem 4.1, this element z is unique and is a tight gate (by symmetry, y is also the unique tight gate such that $\Phi(z) \cap \Phi(y) = \{\beta\}$). □

We also note the following consequence of Theorem 4.1 which shows that the number of non-simple tight gates is always even.

Corollary 6.1. *Let* $|\Gamma^0 \setminus S| = K$. *Then* K *is even, and* $|\mathcal{S} \setminus \Delta| \leq K/2$.

Proof. Theorem 4.1 shows that tight gates $g \in \Gamma^0 \setminus S$ are naturally paired according to their final roots $\Phi^R(g)$ in the following way: For each $\beta \in \mathcal{S} \setminus \Delta$, there is some $x \in \Gamma^0$ with $\beta \in \partial T(x^{-1}) \cap \Phi^R(x)$. Hence by Theorem 4.1 there is a unique element $y \in \Gamma^0 \setminus S$ such that $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^R(y) = \{\beta\}$. By symmetry, x is also the unique element of Γ^0 with $\Phi(x) \cap \Phi(y) = \{\beta\}$. ☐

6.1. **Data for select Coxeter groups.** We utilise Algorithm 6.1 to compute the size of $\mathcal{S}$ and Γ^0 for some Coxeter systems in low rank. We note that as the size of Γ increases, in general, the ratio $|\Gamma^0|/|\Gamma|$ tends to decrease, making the computation of Γ^0 generally much more efficient to determine whether two elements have the same cone type.

W	$\lvert\mathcal{E}\rvert$	$\lvert\mathcal{S}\rvert$	$\lvert L\rvert$	$\lvert L^0\rvert$	$\lvert\Gamma\rvert$	$\lvert\Gamma^0\rvert$
$\widetilde{A}_2$	6	6	16	9	16	9
$\widetilde{B}_2$	8	8	25	14	24	13
$\widetilde{G}_2$	12	12	49	26	41	21
$\widetilde{A}_3$	12	12	125	28	125	28
$\widetilde{B}_3$	18	18	343	66	315	58
$\widetilde{C}_3$	18	18	343	66	317	58
$\widetilde{A}_4$	20	20	1296	75	1296	75
$\widetilde{B}_4$	32	32	6561	270	5789	227
$\widetilde{C}_4$	32	32	6561	270	5860	227
$\widetilde{D}_4$	24	24	2401	140	2400	139
$\widetilde{F}_4$	48	48	28561	1054	22428	715
$\widetilde{A}_5$	30	30	16807	186	16807	186
$\widetilde{B}_5$	50	50	161051	1030	137147	836
$\widetilde{C}_5$	50	50	161051	1030	139457	836
$\widetilde{D}_5$	40	40	59049	608	58965	596

W	$\lvert\mathcal{E}\rvert$	$\lvert\mathcal{S}\rvert$	$\lvert L\rvert$	$\lvert L^0\rvert$	$\lvert\Gamma\rvert$	$\lvert\Gamma^0\rvert$
$X_4(4)$	25	25	438	112	392	98
$X_4(5)$	32	32	516	158	462	138
Y_4	32	32	687	166	578	150
Z_4	30	30	513	142	473	132
$X_5(3)$	114	114	101412	5767	52542	4071
$X_5(4)$	83	83	25708	3128	22886	2871
$X_5(5)$	135	135	42064	6014	37956	5523
Z_5	120	120	41385	5476	39138	5391

FIGURE 5. Data for low rank affine and compact hyperbolic Coxeter groups.

7. Ultra-Low Elements

The remainder of this paper is devoted to proving Theorem 5. Theorem 6 is a consequence of the computations in this section and Section 8.

7.1. **Dihedral groups, right-angled and complete graph Coxeter groups.** We begin with some basic observations and useful results regarding inversion sets of finite Coxeter groups and dihedral groups.

Lemma 7.1. *Let W be a finite Coxeter group and w_o the longest element of W. Then*

$$T(w_o) = \bigcap_{s \in S} H_s^+ = \{e\}$$

and $w_o \in \mathscr{U}$.

Proof. It is clear that $\Phi^1(w_o) = \Delta$ and since $\Phi(w_o) \cap \Phi(s) = \{\alpha_s\}$ for all $s \in S$ we have $\Phi^1(w_o) = \partial T(w_o)$. ☐

Lemma 7.2. *Let $x \in W'$ be ultra-low and W' a standard parabolic subgroup of W. Then x is ultra-low in W.*

Proof. This is clear by the definition of ultra-low elements. ☐

The following technical lemma will be useful for working with finite dihedral groups and shows that $\mathscr{U} = \Gamma = \widetilde{S}$ for these groups. We denote $W_{\langle s,t \rangle}$ to be a dihedral group generated by the reflections s, t. Denote the alternating word $stst\ldots$ of length n by $[s,t]_n$.

Lemma 7.3. *Let $W_{\langle s,t \rangle} = \langle s,t \mid (st)^l = s^2 = t^2 = 1 \rangle$ with $l < \infty$. Then*

(i) *When l is odd,*
   (1) $[s,t]_n(\alpha_t) = [t,s]_{l-1-n}(\alpha_s)$ for n odd and $1 \le n \le l-1$.

(2) $[s,t]_n(\alpha_s) = [t,s]_{l-1-n}(\alpha_t)$ *for* n *even and* $1 \le n \le l-1$.

(ii) *When* l *is even,*

(1) $[s,t]_n(\alpha_t) = [t,s]_{l-1-n}(\alpha_t)$ *for* n *odd and* $1 \le n \le l-1$.

(2) $[s,t]_n(\alpha_s) = [t,s]_{l-1-n}(\alpha_s)$ *for* n *even and* $1 \le n \le l-1$.

(iii) *Let*

$$L = \big(\alpha_s, s(\alpha_t), st(\alpha_s), \dots, [s,t]_{l-1}(\alpha_u)\big)$$

where $u = s$ *for* l *odd and* $u = t$ *for* l *even. Then for* $j \le l-1$

$$\Phi([s,t]_j) = \{\gamma_1, \dots, \gamma_j\}$$
$$\Phi([t,s]_j) = \{\gamma_l, \gamma_{l-1}, \dots, \gamma_{l-j}\}$$

where γ_i *is the ith root in* L. *In particular,*

$$\Phi([s,t]_n) \cap \Phi([t,s]_m) = \{[s,t]_{n-1}(\delta)\} = \{[t,s]_{m-1}(\delta')\}$$

for $n+m = l+1$ *and* $\delta = \alpha_s$ *if* $[s,t]_{n-1}$ *ends in* t *and* $\delta = \alpha_t$ *if* $[s,t]_{n-1}$ *ends in* s *(similarly for* $[t,s]_{m-1}$*) and* δ' *is the other simple root.*

(iv)

$$\Phi^1([s,t]_j) = \{\gamma_1, \gamma_j\} = \partial T([s,t]_j^{-1}) \text{ and} \tag{7.1}$$
$$\Phi^1([t,s]_j) = \{\gamma_l, \gamma_{l-j}\} = \partial T([t,s]_j^{-1}) \qquad \text{for } j \le l-1 \tag{7.2}$$

Proof. $(i)-(iii)$ are straightforward calculations. To prove (iv) consider the following (we discuss the case of (7.1) with (7.2) being entirely similar). The first equality is clear to see since for γ_i with $i \notin \{1, j\}$ we have $\ell(s_{\gamma_i}[s,t]_j) < \ell([s,t]_j) - 1$ and so $\gamma_i \notin \Phi^1([s,t]_j)$ and thus $\gamma_i \notin \partial T([s,t]_j^{-1})$ since $\partial T([s,t]_j^{-1}) \subseteq \Phi^1([s,t]_j)$. It remains to justify that $\Phi^1([s,t]_j) \subset \partial T([s,t]_j^{-1})$. Since γ_1 is simple it is a boundary root. Then the root γ_j is a boundary root since $\gamma_j \in \Phi^R([s,t]_j)$ and $W = \Gamma$ when W is finite, hence the result follows by Theorem 2.3. ☐

Corollary 7.1. *Let* $W_{\langle s,t\rangle} = \langle s,t \mid (st)^l = s^2 = t^2 = 1\rangle$ *with* $l < \infty$ *and denote* w_o *the longest element of* $W_{\langle s,t\rangle}$. *Then*

(i) $\Gamma^0 = W_{\langle s,t\rangle} \setminus \{e, w_o\}$

(ii) $W_{\langle s,t\rangle} = \mathscr{U} = \Gamma = \widetilde{\mathcal{S}}$

(iii) *If* $W_{\langle s,t\rangle} \le W$ *then* $W_{\langle s,t\rangle} \subseteq \mathscr{U}$

Proof. (i) For W a finite dihedral group, we have $W = \Gamma$ and it is clear that $|\Phi^R(w)| = 1$ for all $w \ne e, w_o$. Then (ii) follows by Lemma 7.3 (iv) and Lemma 7.1. Then (iii) follows by Lemma 7.2. ☐

The following result will greatly simplify calculations in this section and shows that any root β whose support $\Gamma(\beta)$ is a standard finite parabolic dihedral root sub system is a boundary root.

Lemma 7.4. *Let* $x \in W$ *and* $\beta \in \Phi^1(x)$ *where* $\Gamma(\beta) = \{s,t\}$ *and* $W' = \langle s,t \mid (st)^l = s^2 = t^2 = 1\rangle$ *is a standard finite parabolic dihedral subgroup. Then there exists a unique* $y \in W'$ *with* $\Phi(x) \cap \Phi(y) = \{\beta\}$ *and* $\Phi^R(y) = \{\beta\}$. *Thus* $\beta \in \partial T(x^{-1})$.

Proof. If β is a simple root the result is clear. Hence assume β is non-simple (note that since $\beta \in \mathscr{E}$ by Lemma 2.4 this then implies that W' is finite). Then $\beta = [s,t]_{n-1}(\delta)$ for some $1 \le n \le l$ and where $\delta = \alpha_s$ if $[s,t]_{n-1}$ ends in t and $\delta = \alpha_t$ if $[s,t]_{n-1}$ ends in s. It then follows from Lemma 7.3 (iv) that

$$\Phi^R([s,t]_n) = \{[s,t]_{n-1}(\delta)\} = \{[t,s]_{m-1}(\delta')\} = \Phi^R([t,s]_m)$$

and $\Phi([s,t]_n) \cap \Phi([t,s]_m) = \{[s,t]_{n-1}(\delta)\}$ where $m+n = l+1$ and δ' is the other simple root. Since $\beta \in \Phi^1(x) \cap \Phi^+_{W'}$ we have $\beta \in \Phi^1(x_J)$ with respect to the unique reduced decomposition $x = x_J \cdot x^J$, where $x_J \in W'$ and $x^J \in X_{W'}$. Then by Lemma 7.3 (iii) it must follow that either $x_J = [s,t]_n$ or $x_J = [t,s]_m$. Without loss of generality, suppose $x_J = [s,t]_n$. We claim that $\Phi(x) \cap \Phi([t,s]_m) = \{\beta\}$. Clearly, $\Phi(x_J) \cap \Phi([t,s]_m) = \{\beta\}$ and since $\Phi(x) = \Phi(x_J) \sqcup x_J \Phi(x^J)$ it remains to justify that $\Phi([t,s]_m) \cap x_J\Phi(x^J) = \emptyset$. If $x^J = e$ then we are done, otherwise for each $\alpha \in x_J\Phi(x^J)$ it follows that $\mathrm{Coeff}_\alpha(\alpha_u) > 0$ for some simple root $\alpha_u \notin \{\alpha_s, \alpha_t\}$. If $x_J = [t,s]_m$ then swapping the roles of $[s,t]_n$ and $[t,s]_m$ in the above argument, we have $\Phi(x) \cap \Phi([s,t]_n) = \{\beta\}$. This completes the proof of our claim. ☐

In [6] the authors provide an explicit description of $\widetilde{\mathcal{S}}$ when W is right-angled (i.e. $m(s,t) \in \{2, \infty\}$ for all $s \neq t$) or when Γ_W is a complete graph (i.e. $m(s,t) \geq 3$ for all $s \neq t$). This description allows us to directly show that these elements are also ultra-low.

Proposition 7.1. [6, Proposition 5.1]

(i) *When W is right-angled then*

$$\widetilde{\mathcal{S}} = \bigcup_{J \subseteq S} W_J$$

where $J \subseteq S$ is a set of pairwise commuting simple reflections.

(ii) *When Γ_W is the complete graph then*

$$\widetilde{\mathcal{S}} = \Big(\cup_{\{(s,t)\in W \mid \ m(s,t)<\infty\}} W_{\langle s,t \rangle} \Big) \bigcup \Big(\cup_{(s,t,r)\in X} \left(t[s,r]_{m(s,r)}\right) \Big)$$

where X consists of all triples $(s,t,r) \in S$ with s,t,r distinct and $m(s,t), m(s,r), m(t,r)$ all finite.

Given the explicit description of $\widetilde{\mathcal{S}}$ in these cases, our goal is then to show that $\widetilde{\mathcal{S}} \subseteq \mathscr{U}$. It then follows from [21, Proposition 6.2] that $\widetilde{\mathcal{S}} \subseteq \mathscr{U} \subseteq \Gamma \subseteq \widetilde{\mathcal{S}}$ and thus $\widetilde{\mathcal{S}} = \mathscr{U} = \Gamma$.

The following relates to Theorem 7.1 (ii).

Lemma 7.5. *Let Γ_W be the complete graph, with notation as in Proposition 7.1. Then*

$$\Phi^1(t[s,r]_{m(s,r)}) = \{\alpha_t, t(\alpha_s), t(\alpha_r)\} = \partial T(t[s,r]^{-1}_{m(s,r)})$$

for r,s,t are distinct and $m(s,t), m(s,r), m(t,r)$ are all finite.

Proof. We prove the first equality. Note that $w_o := [s,r]_{m(s,r)}$ is the longest element of $W_{\langle s,r\rangle}$ and hence $\Phi^1(w_o) = \{\alpha_s, \alpha_r\}$ by Lemma 7.1. The first equality then follows using Proposition 2.2 by direct calculation. The second equality then follows by Lemma 7.4. □

Theorem 7.1. *Let (W,S) be a Coxeter system such that*

(i) W is right-angled, or;

(ii) The Coxeter graph is a complete graph

Then $\mathscr{U} = \Gamma = \widetilde{\mathcal{S}}$.

Proof. (i) By Proposition 7.1, $\widetilde{\mathcal{S}}$ is the union of W_J where $J \subseteq S$ is a set of pairwise commuting reflections. We show that each W_J in this union consists of ultra-low elements. If $|J| = 1$ then the only element of W_J is the simple reflection s_J, which is ultra-low. If $|J| = 2$ then the result follows by Corollary 7.1. For $|J| > 2$ then

$$W_J = \{s_i \mid i \in J\} \cup \{st \mid st = ts \text{ and } s,t \in J\} \cup \{w_o\}$$

where w_o is the product of distinct simple reflections in J (the longest element of W_J). Hence the result follows from Corollary 7.1 and Lemma 7.1.

(ii) In the case of Γ_W being the complete graph, the elements contained in finite dihedral subgroups are ultra-low by Corollary 7.1 and the elements of the form $t[s,r]_{m(s,r)}$ with s,t,r distinct and $m(s,t), m(s,r), m(t,r)$ all finite are ultra-low by Lemma 7.5.

This proves Theorem 5 for the cases (i) and (ii). □

Corollary 7.2. *Let (W,S) be a right-angled Coxeter system. Then $\Gamma^0 = S$.*

Proof. By Theorem 7.1 we have $\widetilde{\mathcal{S}} = \mathscr{U}$ and by Proposition 7.1 if $w \in \widetilde{\mathcal{S}}$ then $w \in W_J$ where $|J| > 2$ is a set of pairwise commuting reflections. It is then clear to see that if $w \notin S$ then $|\Phi^R(w)| > 1$. □

Corollary 7.3. *Let W be a Coxeter group such that Γ_W is a complete graph. Then $\Gamma^0 = \bigcup W_J \setminus \{e, w_J\}$ where W_J is a standard finite dihedral parabolic subgroup of W and w_J is the longest element of W_J.*

Proof. By Theorem 7.1 we have that

$$\mathscr{U} = \Gamma = \widetilde{\mathcal{S}} = \Big(\cup_{\{(s,t)\in W \mid \ m(s,t)<\infty\}} W_{\langle s,t \rangle} \Big) \bigcup \Big(\cup_{(s,t,r)\in X} \left(t[s,r]_{m(s,r)}\right) \Big)$$

where X consists of all triples $(s,t,r) \in S$ with s,t,r distinct and $m(s,t), m(s,r), m(t,r)$ all finite. By Corollary 7.1 the elements in the left side of the above union (with the exception of e and w_J) are tight.

It is then straightforward to verify that the elements of the form $t[s,r]_{m(s,r)}$ are not tight gates. Since $[s,r]_{m(s,r)}\alpha_s = -\alpha_r$ and $[s,r]_{m(s,r)}\alpha_r = -\alpha_s$ it follows by Lemma 2.3 that $t(\alpha_s), t(\alpha_r) \in \Phi^R(t[s,r]_{m(s,r)})$. □

7.2. **Rank 3 irreducible Coxeter groups.** We now discuss the case of irreducible rank 3 Coxeter groups.

When Γ_W is a complete graph, then the result follows by Theorem 7.1. When W is affine of type $\widetilde{B}_2$ or $\widetilde{G}_2$ then $\mathscr{U} = \Gamma$ can be directly verified from the illustration of their cone type arrangements in Figure 4 in the following way: for each gate $g \in \Gamma$ and each hyperplane H separating g from the identity such that $\ell(s_H g) = \ell(g) - 1$ (i.e. the hyperplanes corresponding to the roots $\Phi^1(g)$) there is a tight gate h such that the only hyperplane separating g and h is H. This then directly shows that $\Phi^1(g) = \partial T(g^{-1})$.

Hence we are left with the (non-affine) linear diagrams. We separate these diagrams into three types.

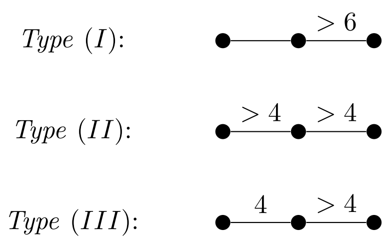


FIGURE 6. Throughout this chapter let the simple reflections s, t, u correspond to the vertices reading left to right and denote $a = m_{s,t}$ and $b = m_{t,u}$. The corresponding cone type automata is illustrated in Figure 7, Figure 8, Figure 9 respectively.

The following result allows us to only consider the above diagrams with all labels finite.

Theorem 7.2. *Let W be a rank 3 Coxeter group with Γ_W a linear diagram and such that one of the standard parabolic dihedral subgroups of W is an infinite dihedral group (i.e. either $W_{\langle s,t\rangle} = W_\infty$ or $W_{\langle t,u\rangle} = W_\infty$ in the labelling convention of Figure 6). Then*

$$W_{\langle s,t\rangle} \cup W_{\langle s,u\rangle} = \Gamma = \widetilde{S} = L = \mathscr{U}$$

.

Proof. By symmetry we may suppose that the right label in one of the diagrams in Figure 6 is ∞, then by [20, Theorem 1], the automaton $\mathcal{A}_{BH}$ is minimal and hence $\mathcal{A}_{\widetilde{S}}$ is minimal since $\widetilde{S} \subseteq L$. Therefore we have $\Gamma = \widetilde{S} = L$.

We claim that $\Gamma = W_{\langle s,t\rangle} \cup W_{\langle s,u\rangle}$. By Corollary 7.1 we have $W_{\langle s,t\rangle} \cup W_{\langle s,u\rangle} \subseteq \Gamma$ and note that by Lemma 2.4 we have $\mathscr{E} = \Phi^+_{\{s,t\}} \sqcup \{\alpha_u\}$. First note that if $w \in W_{\langle t,u\rangle}$ and $w \neq u$ then $\Phi^R(w) = \beta = a\alpha_t + b\alpha_u$ for some $a, b > 0$. Since $\beta \notin \mathscr{E}$, β cannot be a boundary root. Hence by Theorem 2.3 $w \notin \Gamma$. Now if there is $g \in \Gamma$ such that g is not contained in either $W_{\langle s,t\rangle}$ or $W_{\langle s,u\rangle}$ then g must have a suffix, which is also in Γ, in the form of one of the following.

(i) $u[t,s]_k$ for some $1 \leq k \leq m(s,t)$
(ii) $u[s,t]_k$ for some $1 \leq k \leq m(s,t)$
(iii) tu, tsu.

Suppose g has a suffix in the form of (i). Now

$$\partial T([t,s]_k u) \subseteq \Phi^1(u[t,s]_k) \subseteq \{\alpha_u\} \cup u\Phi^1([t,s]_k) = \{\alpha_u\} \cup u\big(\{\alpha_t, c_1\alpha_t + c_2\alpha_s\}\big)$$

for some $c_1, c_2 > 0$, where the first inclusion follows by Lemma 2.6 and the second inclusion follows by Proposition 2.2. The equality then follows by Lemma 7.3. Since $u(\alpha_t) \notin \mathscr{E}$ and $u(c_1\alpha_t + c_2\alpha_s) \notin \mathscr{E}$ it follows that $\partial T([t,s]_k u) = \{\alpha_u\} = \partial T(u)$ and thus $u[t,s]_k$ is not a gate, which contradicts suffix closure. The same argument applies for (ii) and (iii), completing the proof of our claim. Then the fact that $\Gamma = \mathscr{U}$ follows by Corollary 7.1 (iii). ☐

We now provide complete details for the proof of type I and be a little more brief for types II and III since the majority of the proofs are exactly the same (and easier) as in the case of type I and the computations are straightforward. We note that the automata illustrated in Figure 7, Figure 8, and Figure 9 was first studied in [12]. We illustrate them here for the purpose of proving $\mathscr{U} = \Gamma$ and to indicate the tight gates in these groups.

In the remainder of the paper, it will be helpful to record the elementary roots in rank 3 with full support, which can be directly computed following the results in [4].

Lemma 7.6. *Let W be a rank 3 Coxeter group whose graph Γ_W is linear with finite labels (see Figure 6). Let $c_1 = 2\cos\frac{\pi}{m_{st}}$ and $c_2 = 2\cos\frac{\pi}{m_{tu}}$.*

(1) For W of type I. Then

$$\mathscr{E}_S = \{(c_1, c_1, 1), (c_1, c_1, c_1^2 - 1), (1, 1, c_1)\}$$

(2) For W of type II or III then

$$\mathscr{E}_S = \{(c_1, 1, c_2)\}$$

Proof. See [4, Lemma 4.7] and [4, Proposition 6.7]. □

Remark 7.1. Our computations in this section is greatly simplified by Lemma 7.4. For any element x and each $\beta \in \Phi^1(x^{-1})$, if $\Gamma(\beta) \in \Phi^+_{W'}$ for $W' \in \{W_{\langle s,t\rangle}, W_{\langle t,u\rangle}\}$ then we immediately have $\beta \in \partial T(x)$. Thus it suffices to only consider the roots whose support is all of S.

7.3. **Type I.** In this section, we detail the results specific to Type I graphs.

Lemma 7.7. *Let W be of type I. Then*

(i) $T(st) \xrightarrow{u} T(stu)$
(ii) $T(sts) \xrightarrow{u} T(stsu)$
(iii) $T(stsu) \xrightarrow{t} T(stsut)$
(iv) $T(stsut) \xrightarrow{u} T(stsutu)$

are transitions in $\mathcal{A}_o$, and each element above is ultra-low.

(Note that this result corresponds to transitions leaving nodes 15, 16, 20, 26 *in Figure 7)*

Proof. (i) By Lemma 7.3 (iv) we have $\Phi^1(ts) = \partial T(st) = \{\alpha_t, s\alpha_t\}$ and using Proposition 2.2 we compute $\Phi^1(uts) = \{\alpha_u, u\alpha_t, ut\alpha_s\}$ (note that by the calculations in Section A.1 uts is a tight gate with $\Phi^R(uts) = \{ut\alpha_s\}$ and so $ut\alpha_s \in \partial T(stu)$). It then follows by Theorem 4.1 and the calculations in Section A.1 that we must have

$$\Phi^1(uts) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s = (1, 1, c_1)$$

(ii) By Lemma 7.1 we have $\Phi^1(sts) = \{\alpha_s, \alpha_t\}$ and using Proposition 2.2 we compute $\Phi^1(usts) = \{\alpha_s, \alpha_u, u\alpha_t\}$. So the result is immediate by Lemma 7.4.

(iii) We compute $\Phi^1(tusts) = \{\alpha_t, t\alpha_s, tust\alpha_s\}$. Note that $tust\alpha_s = tu\alpha_t$, then again we are done by Lemma 7.4.

(iv) Finally, we compute $\Phi^1(utusts) = \{\alpha_u, utu\alpha_t, ut\alpha_s\}$. We claim that

$$\Phi^1(utusts) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s = (1, 1, c_1)$$

It suffices to show that $\alpha_u, utu\alpha_t \notin \Phi(s[t,u]_{b-1})$. Since $s \notin D_L([t,u]_{b-1})$ we have $\Phi(s[t,u]_{b-1}) = \{\alpha_s\} \sqcup s\Phi([t,u]_{b-1})$ and hence $\mathrm{Coeff}_{\alpha_s}(\beta) > 0$ for each $\beta \in \Phi(s[t,u]_{b-1})$. □

For the remaining results we will use Lemma 7.4 without further direct reference.

Lemma 7.8. *Let W be of type I. Then*

(i) $T(su) \xrightarrow{t} T(sut)$
(ii) $T(sut) \xrightarrow{u} T(sutu)$

and each element above is ultra-low.

(Note that this result corresponds to the transitions leaving the nodes 18, 19 *in Figure 7).*

Proof. For (i), su is ultra-low by Corollary 7.1 and we compute $\Phi^1(tus) = \{\alpha_t, t\alpha_u, t\alpha_s\}$, so the result is immediate.

For (ii) we compute from (i) to obtain $\Phi^1(utus) = \{\alpha_u, ut\alpha_u, ut\alpha_s\}$. By the same argument as in Lemma 7.7 (iv), we have $\Phi(utus) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s$. □

Lemma 7.9. *Let W be of type I with $k = b - 2$.*

(i) *If b is odd, then $T([t,u]_k) \xrightarrow{s} T([t,u]_k s)$*
(ii) *If b is even, then $T([u,t]_k) \xrightarrow{s} T([u,t]_k s)$*

and the elements above are ultra-low.

(Note that this result corresponds to the s-transition from the nodes 9 *to* 23 *in Figure 7).*

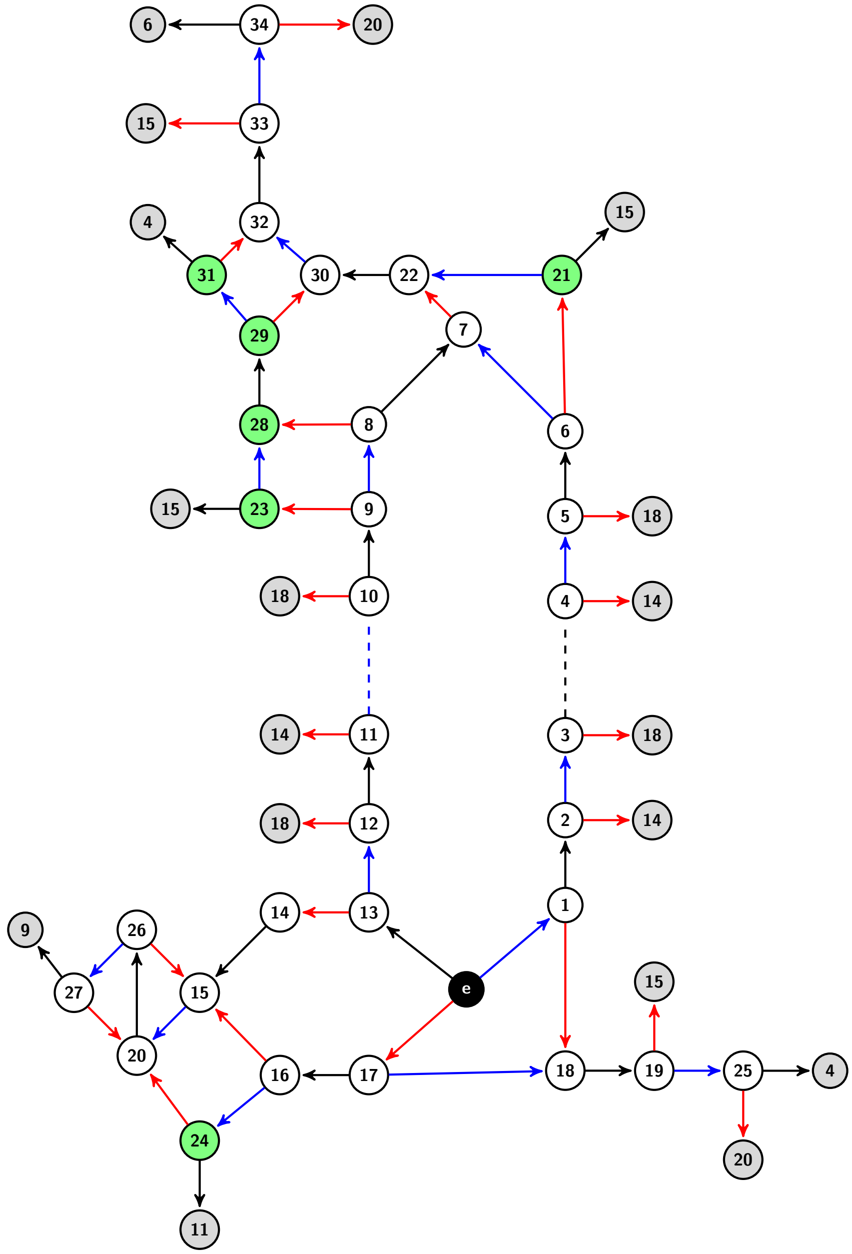


FIGURE 7. The automaton $\mathcal{A}_o$ for (W, S) of Type I with b odd (the illustration includes all states for the case $b = 7$ with the trailing blue and black dots indicating the additional elements of $W_{\langle t,u\rangle}$ when $b > 7$). The node coloured black denotes the identity (start state) and the red, black and blue arrows correspond to s, t and u transitions respectively. Transitions into gray filled nodes informs the reader to continue the reading of a word by continuing the path at the corresponding white filled node with the same label. The nodes highlighted green are the inverses of the tight gates whose final root has non-spherical support (see Section A).

Proof. The proof is entirely the same for cases (i) and (ii), so we will just prove (i). Note that in both cases, the alternating words $[t, u]_k$ and $[u, t]_k$ end in t and by Lemma 7.3 (iv) we have that

$$\Phi^1([u, t]_k^{-1}) = \Phi^1([t, u]_k) = \{\alpha_t, ut\alpha_u\}$$

We then compute $\Phi^1(s[u,t]_k^{-1}) = \Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$. Note that $sut\alpha_u = (c_1, c_1, c_1^2 - 1)$. We claim that

$$\Phi^1(s[t,u]_k) \cap \Phi(utus[t,u]_{b-2}) = \{sut\alpha_u\}$$

Since $utus[t,u]_{b-2}$ is reduced, we have $\Phi(utus[t,u]_{b-2}) = \Phi(utus) \sqcup utus\Phi([t,u]_{b-2})$. Clearly, $sut\alpha_u \in \Phi(utus)$ and $\{\alpha_s, s\alpha_t\} \not\subset \Phi(utus)$. Then note that $\mathrm{Coeff}_{\alpha_u}(\beta) > 0$ for $\beta \in utus\Phi([t,u]_{b-2})$, hence $\alpha_s, s\alpha_t \notin \Phi(utus[t,u]_{b-2})$. □

Lemma 7.10. *Let W be of type I with $k = b - 2$.*

(i) *If b is odd then $T([t,u]_k s) \xrightarrow{u} T([t,u]_k su)$*
(ii) *If b is even then $T([u,t]_k s) \xrightarrow{u} T([u,t]_k su)$*

and the elements above are ultra-low.

(Note that this result corresponds to the u-transition from the nodes 23 to 28 in Figure 7).

Proof. The argument for both cases is again identical and so we will prove (i) (the proof of (ii) is exactly the same, by replacing $[t,u]_k$ with $[u,t]_k$).

In Lemma 7.9 we computed $\Phi^1(s[t,u]_k)$, and hence we compute $\Phi^1(us[t,u]_k) = \{\alpha_u, \alpha_s, st\alpha_u\}$. We show that $st\alpha_u = (c_1, c_1, 1)$ is a boundary root of $T([t,u]_k su)$. We claim that $\Phi^1(us[t,u]_k) \cap \Phi(tus[t,u]_k) = \{(c_1, c_1, 1)\}$. We have $\Phi(tus[t,u]_k) = \Phi(tus) \sqcup tus\Phi([t,u]_k)$. Clearly, $(c_1, c_1, 1) \in \Phi(tus)$ and since $tus[t,u]_k$ is a reduced expression and is a tight gate by Section A.1 it follows that $\alpha_u, \alpha_s \notin \Phi(tus[t,u]_k)$. □

Lemma 7.11. *Let W be of type I with $b = m_{ut}$ odd and let $k = b - 2$. Then*

(i) $T([t,u]_k us) \xrightarrow{t} T([t,u]_k ust)$
(ii) $T([t,u]_k ust) \xrightarrow{s} T([t,u]_k usts)$
(iii) $T([t,u]_k ust) \xrightarrow{u} T([t,u]_k ustu)$
(iv) $T([t,u]_k ustu) \xrightarrow{s} T([t,u]_k ustus)$
(v) $T([t,u]_k ustus) \xrightarrow{t} T([t,u]_k ustust)$
(vi) $T([t,u]_k ustust) \xrightarrow{u} T([t,u]_k ustustu)$

and the elements above are ultra-low.

When b is even, replace $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the transitions leaving the nodes (28, 29, 30, 31, 32, 33 and 34) in Figure 7).

Proof. Again, the case for b even is entirely the same and hence we show the case of b odd. For b even, replace $[t,u]_k$ with $[u,t]_k$ in the below arguments.

(i) From Lemma 7.10 we computed $\Phi^1(su[t,u]_k)$, hence we compute $\Phi^1(tsu[t,u]_k) = \{\alpha_t, t\alpha_u, t\alpha_s, st\alpha_u\}$. By Section A the element $tsu[t,u]_k$ is a tight gate and then by Theorem 4.1 we have $\Phi^1(tsu[t,u]_k) \cap \Phi(tsu) = \{st\alpha_u\}$.
(ii) We then compute $\Phi^1(stsu[t,u]_k) = \{\alpha_s, \alpha_t, t\alpha_u\}$.
(iii) Compute $\Phi^1(utsu[t,u]_k) = \{\alpha_u, ut\alpha_u, sut\alpha_u\}$. Then note from Lemma 7.9 (ii) we had

$$\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$

Hence $\Phi^1(utsu[t,u]_k) \cap \Phi(s[t,u]_k) = \{sut\alpha_u\}$.
(iv) We compute $\Phi^1(sutsu[t,u]_k) = \{\alpha_s, \alpha_u, ut\alpha_u\}$. So the result is immediate and similarly;
(v) We compute $\Phi^1(tsutsu[t,u]_k) = \{\alpha_t, t\alpha_s, tut\alpha_u\}$.
(vi) Finally, compute $\Phi^1(utsutsu[t,u]_k) = \{\alpha_u, ut\alpha_s, utut\alpha_u\}$. And by the computations in Lemma 7.7 (i) we have $\Phi^1(utsutsu[t,u]_k) \cap \Phi(s[t,u]_{b-1}) = \{ut\alpha_s\}$.

□

Lemma 7.12. *Let W be of type I with $b = m_{ut}$ odd and $k = b - 1$. Then*

(i) $T(w_{ut}) \xrightarrow{s} T(w_{ut}s)$
(ii) $T([u,t]_k) \xrightarrow{s} T([u,t]_k s)$

and the elements above are ultra-low.

When b is even, replace $[u,t]_k$ with $[t,u]_k$ in (ii), then both results hold.

(Note that this result corresponds to the s-transitions leaving the nodes 7 and 6 in Figure 7).

Proof. By Lemma 7.1 we have that w_{ut} is ultra-low then compute $\Phi^1(sw_{ut}) = \{\alpha_u, \alpha_s, \alpha_s + \alpha_t\}$ and hence the result is immediate.

Similarly, we have $\Phi^1([t,u]_k) = \{\alpha_t, u\alpha_t\}$ and compute

$$\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$

Then by Lemma 7.11 (iii) we have $\Phi^1(s[u,t]_k) \cap \Phi(utus[t,u]_{b-2}) = \{sut\alpha_u\}$. □

7.3.1. *Remaining Transitions.* The previous results together show that the inverses of the minimal length representatives of the states (gates of $\mathscr{T}$) coloured white and green in Figure 7 are indeed ultra-low. The final step is then to show that there are not any more gates. The following sets of results prove the transitions to the gray nodes in Figure 7.

Lemma 7.13. *Let (W,S) be a Coxeter system whose graph is contained in Figure 6 and let $b = m_{ut}$.*

(1) For W of Type I:
- (i) *If b is odd, let w be an element of the form:*
  - *(a) $[t,u]_n$ for $2 \le n \le b-3$ for n even.*
  - *(b) $[u,t]_n$ for $3 \le n \le b-2$ for n odd.*
- (ii) *If b is even, let w be an element of the form:*
  - *(a) $[t,u]_n$ for $2 \le n \le b-2$ for n even.*
  - *(b) $[u,t]_n$ for $3 \le n \le b-3$ for n odd.*

Then $T(w) \to T(us)$ is an s-transition in $\mathcal{A}_o$.

(Note that this result corresponds to the s-transitions leaving the nodes, $3, 5, 10, 12$ *in Figure 7 and* $13, 16, 2, 4, 7$ *in Figure 8).*

Proof. Note that in each case, the elements $[t,u]_n$ and $[u,t]_n$ end in u. Clearly $\ell(ws) > \ell(w)$ so ws is reduced. We now show $T(us) = T(ws)$. We prove (i) with (ii) being entirely the same argument. Hence let b be odd and $n = b-2$. Note that this means $[u,t]_n = [u,t]_n^{-1}$. By Lemma 7.3 (iv) we have

$$\Phi^1(w) = \Phi^1([u,t]_n) = \partial T([u,t]_n) = \{\alpha_u, tu\alpha_t\}$$

Using Proposition 2.2 we compute $\Phi^1(sw) = \{\alpha_s, \alpha_u, stu\alpha_t\}$. By direct calculation it follows that $stu\alpha_t \notin \mathscr{E}$ (since the coefficient of α_s in $stu\alpha_t$ is $c_1^2 - 1$ in the notation of Lemma 7.6. Thus by Lemma 2.6, $stu\alpha_t \notin \partial T(ws)$. Since simple roots are always boundary roots, we have $\partial T(ws) = \{\alpha_s, \alpha_u\} = \partial T(us)$. Therefore by Lemma 2.8 we have the following chain of inclusions:

$$T(us) \subseteq T([u,t]_n s) \subseteq T([t,u]_{n-1} s) \subseteq \ldots \subseteq T(us)$$

which includes all the elements in (a) and (b). □

Lemma 7.14. *Let (W,S) be a Coxeter system of type (I) and let $b = m_{ut}$.*

- (i) *If b is odd, let w be an element of the form:*
  - *(1) $[t,u]_n$ for $3 \le n \le b-4$ for n odd.*
  - *(2) $[u,t]_n$ for $2 \le n \le b-3$ for n even.*
- (ii) *If b is even, let w be an element of the form:*
  - *(1) $[t,u]_n$ for $3 \le n \le b-3$ for n odd.*
  - *(2) $[u,t]_n$ for $2 \le n \le b-4$ for n even.*

then $T(w) \to T(ts)$ is a transition in $\mathcal{A}_o$.

(We note that this result corresponds s-transitions leaving the nodes, $2, 4, 11$ *in Figure 7).*

Proof. The proof is the same as in Lemma 7.13. We again prove (i) with (ii) being the exact same calculation. Let $n = b-3$, so $[u,t]_n = [t,u]_n^{-1}$ then by Lemma 7.3 (iv) we have $\partial T([u,t]_n) = \Phi^1([t,u]_n) = \{\alpha_t, tut\alpha_u\}$. Using Proposition 2.2, we compute $\Phi^1(s[t,u]_n) = \{\alpha_s, s\alpha_t, stut\alpha_u\}$. Since $stut\alpha_u \notin \mathscr{E}$ (which can be seen from Lemma 7.6), because the coefficient of α_s is not c_1 or 1) it follows by Lemma 2.6 that $stut\alpha_u \notin \partial T([u,t]_n s)$. We can then immediately conclude that $\partial T([u,t]_n s) = \{\alpha_s, s\alpha_t\} = \partial T(ts)$ and using Lemma 2.8 we have the chain of inclusions

$$T(ts) \subseteq T([u,t]_n s) \subseteq T([t,u]_{n-1} s) \subseteq \ldots \subseteq T(ts)$$

which includes all the elements in (a) and (b). □

Lemma 7.15. *Let W be of type I with $b = m_{ut}$ odd and $k = b-1$. Then*

- (i) $T([t,u]_{k-1} s) \xrightarrow{t} T(sts)$

(ii) $T([t,u]_{k-1}ustust) \xrightarrow{s} T(sts)$
(iii) $T(stsut) \xrightarrow{s} T(sts)$
(iv) $T([u,t]_k s) \xrightarrow{t} T(sts)$

When b is even, swap $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the transitions to the gray coloured nodes labelled 15 from the nodes 19, 21, 23, 33 and the s-transition from 26 to 15 in Figure 7).

Proof. Cases (i)-(iv) are all similar. Again, swap $[t,u]_k$ with $[u,t]_k$ in all calculations below for b even.

From Lemma 7.9 we have $\Phi^1(s[t,u]_{k-1}) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$. Then using Proposition 2.2 we compute $\Phi^1(ts[t,u]_{k-1}) = \{\alpha_t, \alpha_s, tsut\alpha_u\}$. However, by Lemma 7.6,we have $tsut\alpha_u \notin \mathscr{E}$ and thus by the fact that simple roots are always boundary roots, it follows that $\partial T([t,u]_k st) = \{\alpha_s, \alpha_t\} = \partial T(sts)$.

(ii), we have

$$\Phi^1(tsutsu[t,u]_{k-1}) = \{\alpha_t, t\alpha_s, tut\alpha_u\}$$

and

$$\Phi^1(stsutsu[t,u]_{k-1}) = \{\alpha_s, \alpha_t, stut\alpha_u\}$$

Now $stut(\alpha_u) \notin \mathscr{E}$ and hence $\partial T([t,u]_k ustust) = \{\alpha_s, \alpha_t\} = \partial T(sts)$.

(iii) we computed in Lemma 7.7 that $\Phi^1(tusts) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$. Again, it can be directly checked from Lemma 7.6 that $stut\alpha_t \notin \mathscr{E}$. Therefore, $\partial T(stsuts) = \{\alpha_s, \alpha_t\}$.

(iv) From Lemma 7.12 we computed $\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, ut\alpha_s\}$ and we now compute $\Phi^1(ts[t,u]_k) = \{\alpha_s, \alpha_t, tut\alpha_s\}$. Again $tut\alpha_s \notin \mathscr{E}$, so $\partial T([u,t]_k st) = \{\alpha_s, \alpha_t\}$. ☐

There are just a few remaining cases for W of type I.

Lemma 7.16. *Let W be of type I with $b = m_{ut}$ odd and $k = \ell(w_{ut}) - 2$. Then*

(i) $T([t,u]_k sutu) \xrightarrow{t} T(utut)$
(ii) $T([t,u]_k ustustu) \xrightarrow{s} T(stsu)$
(iii) $T([t,u]_k sutustu) \xrightarrow{t} T([u,t]_6)$
(iv) $T(w_{ut}s) \xrightarrow{t} T([t,u]_k usts)$
(v) $T(stu) \xrightarrow{t} T(tut)$
(vi) $T(stsutu) \xrightarrow{t} T(tutut)$
(vii) $T(sutu) \xrightarrow{t} T(utut)$
(viii) $T(sutu) \xrightarrow{s} T(stsu)$

When b is even, swap $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the following transitions in Figure 7: $31 \to 4$, $34 \to 20$, $34 \to 6$, $22 \to 30$, $24 \to 11$, $27 \to 9$, $25 \to 4$ and $25 \to 20$).

Proof. Again, swap $[t,u]_k$ with $[u,t]_k$ in all calculations below for b even.

(i) We have from Lemma 7.11 (iii) that

$$\Phi^1(utsu[t,u]_k) = \{\alpha_u, ut\alpha_u\, sut\alpha_u\}.$$

Then using Proposition 2.2 we compute

$$\Phi^1(tutsu[t,u]_k) = \{\alpha_t, tut\alpha_u, tsut\alpha_u\}$$

Since $tsut\alpha_u \notin \mathscr{E}$ by Lemma 7.4 we immediately have

$$\partial T([t,u]_k ustu) = \{\alpha_t, tut\alpha_u\} = \partial T(utut).$$

(ii) From Lemma 7.11 (vi) we had

$$\Phi^1(utsutsu[t,u]_k) = \{\alpha_u, ut\alpha_s, utut\alpha_u\}$$

and thus compute $\Phi^1(sutsutsu[t,u]_k) = \{\alpha_s, \alpha_u, u\alpha_t, sutut\alpha_u\}$. Then since $sutut\alpha_u \notin \mathscr{E}$ by Lemma 7.7 (ii) and Lemma 7.4 again, we have

$$\partial T([t,u]_k sutustsu) = \{\alpha_u, \alpha_s, u\alpha_t\} = \partial T(stsu)$$

(iii) Following from (ii) we compute $\Phi^1(tutsutsu[t,u]_k) = \{\alpha_t, tut\alpha_s, tutut\alpha_u\}$. Note that $tut\alpha_s \notin \mathscr{E}$ so

$$\partial T([t,u]_k sutustut) = \{\alpha_t, tutut\alpha_u\} = \partial T([u,t]_6).$$

(iv) From Lemma 7.12 (i) we had $\Phi^1(sw_{ut}) = \{\alpha_u, \alpha_s, \alpha_s + \alpha_t\}$ and thus we compute $\Phi^1(tsw_{u,t}) = \{\alpha_t, \alpha_s, t\alpha_u\} = \partial T([t,u]_k usts)$ from Lemma 7.11 (ii).

(v) From Lemma 7.7 (i) we have $\Phi^1(uts) = \{\alpha_u, u\alpha_t, ut\alpha_s\}$. Then using Proposition 2.2 we compute $\Phi^1(tuts) = \{\alpha_t, tu\alpha_t, tut\alpha_s\}$. Since $tut\alpha_s \notin \mathscr{E}$, we must have $\partial T(stut) = \{\alpha_t, tu(\alpha_t)\} = \partial T(tut)$.

(vi) Similarly, from Lemma 7.7 (iv) we have $\Phi^1(utusts) = \{\alpha_u, utu\alpha_t, ut\alpha_s\}$. Then we compute $\Phi^1(tutusts) = \{\alpha_t, tutu\alpha_t, tut(\alpha_s)\}$ and $tut(\alpha_s) \notin \mathscr{E}$ so $\partial T(stsutut) = \{\alpha_t, tutu\alpha_t\} = \partial T(tutut)$.

(vii) From Lemma 7.8 (ii) we have $\Phi^1(utus) = \{\alpha_u, ut\alpha_u, ut\alpha_s\}$. Then we compute $\Phi^1(tutus) = \{\alpha_t, tut\alpha_u, tut\alpha_s\}$. Again $tut\alpha_s \notin \mathscr{E}$ and the result follows.

(viii) Finally compute $\Phi^1(sutus) = \{\alpha_s, \alpha_u, sut\alpha_u, u\alpha_t\}$. Note that $stsu$ is a suffix of $sutus = ustsu$, so $T(sutus) \subseteq T(stsu)$, and by Lemma 7.7 (ii) we have $\partial T(stsu) = \{\alpha_s, \alpha_u, u\alpha_t\}$. To show equality, we just need to show that $sut\alpha_u \notin \partial T(sutus)$. Note that $sut\alpha_u = (c_1, c_1, c_1^2 - 1)$. By Theorem 4.1 and Section A.1, if $sut\alpha_u \in \partial T(sutus)$, then we must have either $\Phi(sutus) \cap \Phi(s[t,u]_{b-2}) = (c_1, c_1, c_1^2 - 1)$ or $\Phi(sutus) \cap \Phi(utus[t,u]_{b-2}) = (c_1, c_1, c_1^2 - 1)$. But clearly, $\alpha_s \in \Phi^1(s[t,u]_{b-2})$ and $\alpha_u \in \Phi^1(utus[t,u]_{b-2})$. Therefore, $sut\alpha_u \notin \partial T(sutus)$.

□

The above results complete the proof for W of type I.

Theorem 7.3. *Let (W, S) be the Coxeter system of Type I with Γ_W as illustrated in Figure 6. Let a be the left edge label and b the right edge label in Γ_W. Let W' be the Coxeter system obtained from W by replacing b with $b+1$ in Γ_W and denote $\mathscr{U}_{W'}$ and $\mathscr{U}_W$ to be the ultra-low elements respectively. Then*

$$|\mathscr{U}_{W'}| = |\mathscr{U}_W| + 2$$

The effect of increasing the label b by 1 adds only two additional ultra low elements corresponding to the two additional elements of the finite dihedral group $W_{\langle t,u\rangle}$. The automaton is illustrated in Figure 7 for b odd (the automaton for b even is entirely similar with only a swapping of the alternating word $[t,u]$ with $[u,t]$ in some instances).

Corollary 7.4. *Let W be of type I with $7 \leq b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 35 + 2(b-7) = 21 + 2b$$

7.4. **Type II.** We illustrate the cone type automaton in Figure 8 below. A number of results in this section apply to both Types II and III.

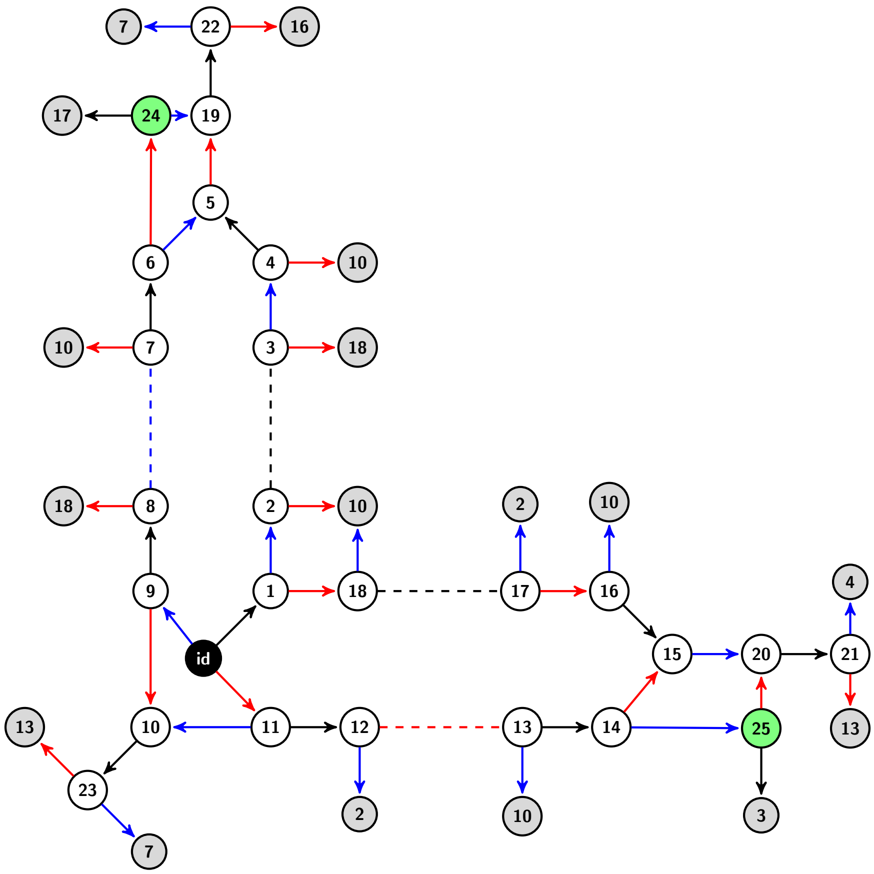


FIGURE 8. The automaton $\mathcal{A}_o$ for (W, S) of Type II with a and b odd (the illustration includes all states for the case $a = b = 5$ with the trailing dashes indicating additional elements of $W_{\langle t,u\rangle}$ and $W_{\langle t,s\rangle}$ when $a, b > 5$). One may note that the automaton here is similar to Type I but less complicated in the sense that there are less nodes at the top of the figure. Again, the states highlighted green indicate the (inverses of) tight gates whose final root has non-spherical support.

Lemma 7.17. *Let W be of type II or III. Then $T(su) \xrightarrow{t} T(sut)$ is a transition in $\mathcal{A}_o$ and sut is ultra-low. (Note that this result corresponds to the transition $10 \to 23$ in Figure 8 and $10 \to 22$ in Figure 9).*

Proof. The proof is exactly the same computation as in Lemma 7.8 (i). □

Lemma 7.18. *Let W be of type II with a, b odd. Then*

(i) $T([s,t]_{a-1}) \xrightarrow{u} T([s,t]_{a-1}u)$
(ii) $T(w_{st}) \xrightarrow{u} T(w_{st}u)$
(iii) $T(w_{st}u) \xrightarrow{t} T(w_{st}ut)$

and

(iv) $T([u,t]_{b-1}) \xrightarrow{s} T([u,t]_{b-1}s)$
(v) $T(w_{ut}) \xrightarrow{s} T(w_{ut}s)$
(vi) $T(w_{ut}s) \xrightarrow{t} T(w_{ut}st)$

where each element w above is ultra-low.

If a is even then swap $[s,t]_{a-1}$ in (i) with $[t,s]_{a-1}$ and if b is even then swap $[u,t]_{b-1}$ in (iv) with $[t,u]_{a-1}$, then the same results hold.

(Note that this result corresponds to the transitions $14 \to 25, 15 \to 20, 20 \to 21, 6 \to 24, 5 \to 19, 19 \to 22$, in Figure 8).

Proof. Note that (iv)-(vi) is the same as (i)-(iii) with the roles of s and u swapped, hence we will just provide the details for (i)-(iii). If a is even, then replace $[s,t]_{a-1}$ with $[t,s]_{a-1}$ in the proofs below, and the same calculations hold. Similarly if b is even, replace $[u,t]_{b-1}$ with $[t,u]_{b-1}$.

(i) By Lemma 7.3 we have $\Phi^1([t,s]_{a-1}) = \{\alpha_t, s\alpha_t\}$ and thus using Proposition 2.2 we compute $\Phi^1(u[t,s]_{a-1}) = \{\alpha_u, u\alpha_t, us\alpha_t\}$. By Lemma 7.4 again we just need to justify that $us\alpha_t = (c_1, 1, c_2) \in \partial T([s,t]_{a-1}u)$. Now $\Phi^1([t,u]_{b-1}) = \{\alpha_t, u(\alpha_t)\}$ and by Proposition 2.2 we compute $\Phi^1(s[t,u]_{b-1}) = \{\alpha_s, s\alpha_t, us\alpha_t\}\}$. Therefore,
$$\Phi^1(u[t,s]_{a-1}) \cap \Phi(s[t,u]_{b-1}) = \{us\alpha_t\}$$

(ii) By Lemma 7.1 we have $\Phi(w_{st}) = \{\alpha_s, \alpha_t\}$ and compute
$$\Phi^1(uw_{st}) = \{\alpha_u, \alpha_s, u\alpha_t\}$$
hence the result by Lemma 7.4.

(iii) We compute $\Phi^1(tuw_{st}) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$, and hence the result.

□

This proves that each of the states indicated by black outlined nodes in Figure 8 are ultra-low. We now prove that there are not any more (and prove the remaining transitions into the gray coloured states).

Lemma 7.19. *Let W be of type II with a, b odd. If w be an element of the form*

(i) *(1) $[t,s]_n$ for $2 \leq n \leq a-1$ for n even*
(2) $[s,t]_n$ for $3 \leq n \leq a-2$ for n odd

Then $T(w) \xrightarrow{u} T(us)$ is a transition in $\mathcal{A}_o$, and if w is an element of the form;

(ii) *(1) $[t,u]_n$ for $2 \leq n \leq b-1$ for n even*
(2) $[u,t]_n$ for $3 \leq n \leq b-2$ for n odd

Then $T(w) \xrightarrow{s} T(us)$ is a transition in $\mathcal{A}_o$.

If a is even then (1) and (2) in (i) should be replaced with
(1) $[s,t]_n$ for $3 \leq n \leq a-1$ for n odd
(2) $[t,s]_n$ for $2 \leq n \leq a-2$ for n even

Similarly if b is even then (1) and (2) in (ii) should be replaced with
(1) $[u,t]_n$ for $3 \leq n \leq b-1$ for n odd
(2) $[t,u]_n$ for $2 \leq n \leq b-2$ for n even

(Note that this result corresponds to the u and s transitions from the nodes labelled 13, 16, 18, 2, 4, 7 *in Figure 8).*

Proof. The proof is the same Lemma 7.13. We again prove (i) in the case of a odd and omit the details for (ii).

Let $n = a-1$, then $\Phi^1([t,s]_n^{-1}) = \{\alpha_s, t\alpha_s\}$. Using Proposition 2.2 we compute
$$\Phi^1(u[t,s]_n^{-1}) = \{\alpha_u, \alpha_s, ut\alpha_s\}$$
then since $ut\alpha_s \notin \mathscr{E}$, we have $\partial T([t,s]_n u) = \{\alpha_u, \alpha_s\} = \partial T(us)$. Then by Lemma 2.8 we have the chain of inclusions
$$T(us) \subseteq T([t,s]_n u) \subseteq T([s,t]_{n-1}u) \subseteq \ldots \subseteq T(tsu) \subseteq T(us)$$
which includes all the elements in (1) and (2). □

The next result is very similar to Lemma 7.19.

Lemma 7.20. *Let W be of type II. When b is odd and w is an element of the form*

(1) $[t,u]_n$ *for* $3 \leq n \leq b-2$ *for* n *odd*
(2) $[u,t]_n$ *for* $2 \leq n \leq b-3$ *for* n *even*

Then $T(w) \xrightarrow{s} T(ts)$ *is a transition in* $\mathcal{A}_o$. *If* b *is even then* (1) *and* (2) *above should be replaced with*

(1) $[u,t]_n$ *for* $2 \leq n \leq b-2$ *for* n *even*
(2) $[t,u]_n$ *for* $3 \leq n \leq b-3$ *for* n *odd*

(Note that this result corresponds to the s-transitions from the nodes labelled $3, 8$ *in Figure 8). When a is odd and w is an element of the form*

(1) $[t,s]_n$ *for* $3 \leq n \leq a-2$ *for* n *odd*
(2) $[s,t]_n$ *for* $2 \leq n \leq a-3$ *for* n *even*

Then $T(w) \xrightarrow{u} T(tu)$ *is a transition in* $\mathcal{A}_o$. *If* a *is even then* (1) *and* (2) *above should be replaced with*

(1) $[s,t]_n$ *for* $2 \leq n \leq a-2$ *for* n *even*
(2) $[t,s]_n$ *for* $3 \leq n \leq a-3$ *for* n *odd*

(Note that this result corresponds to the u-transitions from the nodes labelled $12, 17$ *in Figure 8).*

Proof. The proof is essentially the same as Lemma 7.19. We again show the case of b odd and leave the computation for the other cases to the reader. Let $n = b-2$, then $\Phi^1([t,u]_n^{-1}) = \Phi^1([t,u]_n) = \{\alpha_t, ut\alpha_u\}$. Using Proposition 2.2 we compute

$$\Phi^1(s[t,u]_n) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$

then since $sut\alpha_u \notin \mathscr{E}$, we have $\partial T([t,u]_n s) = \{\alpha_s, s\alpha_t\} = \partial T(ts)$. Then by Lemma 2.8 we have the chain of inclusions

$$T(ts) \subseteq T([t,u]_n s) \subseteq T([u,t]_{n-1}s) \subseteq \ldots \subseteq T(uts) \subseteq T(ts)$$

which includes all the elements in (1) and (2). □

The final result of this section covers the remaining transitions.

Lemma 7.21. *Let W be of type II with a, b odd. Then*

(i) $T([s,t]_{a-1}u) \xrightarrow{t} T(tut)$
(ii) $T(w_{st}ut) \xrightarrow{u} T(tutu)$
(iii) $T(w_{st}ut) \xrightarrow{s} T(sts)$
(iv) $T([u,t]_{b-1}s) \xrightarrow{t} T(tst)$
(v) $T(w_{ut}st) \xrightarrow{u} T(utu)$
(vi) $T(w_{ut}st) \xrightarrow{s} T(tsts)$
(vii) $T(ust) \xrightarrow{s} T(sts)$
(viii) $T(ust) \xrightarrow{u} T(utu)$

If a is even then swap $[s,t]_{a-1}$ *in (i) with* $[t,s]_{a-1}$ *and if b is even then swap* $[u,t]_{b-1}$ *in (iv) with* $[t,u]_{b-1}$, *then the same results hold.*

(Note that this result corresponds to the transitions from the nodes labelled $25 \to 3, 21 \to 4, 21 \to 13, 24 \to 17, 22 \to 7, 22 \to 16, 23 \to 13, 23 \to 7$ *in Figure 9).*

Proof. Again, if a is even, then replace $[s,t]_{a-1}$ with $[t,s]_{a-1}$ in the proofs below, and the same calculations hold. Similarly if b is even, replace $[u,t]_{b-1}$ with $[t,u]_{b-1}$.

(i) From Lemma 7.19 we have $\Phi^1(u[t,s]_{a-1}) = \{\alpha_u, u\alpha_t, us\alpha_t\}$ and compute $\Phi^1(tu[t,s]_{a-1}) = \{\alpha_t, tu\alpha_t, tus\alpha_t\}$. Then since $tus\alpha_t \notin \mathscr{E}$ it follows by Lemma 7.4 that we must have

$$\partial T([s,t]_{a-1}ut) = \{\alpha_t, tu(\alpha_t)\} = \partial T(tut)$$

(ii) By Lemma 7.18 (iii) we have $\Phi^1(tuw_{st}) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$. Then we compute $\Phi^1(utuw_{st}) = \{\alpha_u, ut\alpha_s, utu\alpha_t\}$. Again since $ut\alpha_s) \notin \mathscr{E}$ the result follows.

The proofs of (iii)-(viii) again all have the exact same structure and can be easily verified. For each element w on the left hand side of the arrow, we compute $\Phi^1(s_i w)$. Then there is a single $\beta \in \Phi^1(s_i w) \setminus \mathscr{E}$ with

$\beta \in \Phi^R(s_i w)$. Hence $\beta \notin \partial T(w^{-1}s_i)$ by Lemma 2.6. The remaining roots of $\Phi^1(s_i w)$ have dihedral support and therefore are boundary roots by Lemma 7.4. □

This completes the case for W of type II. We have the following consequence combining the results in this section.

Theorem 7.4. *Let (W,S) be the Coxeter system of Type II with Γ_W as illustrated in Figure 6. Let a be the label of the left edge and b the label of the right edge in Γ_W. Let W' be the Coxeter system obtained from W by replacing b with $b+1$ or by replacing a with $a+1$ in Γ_W. Then*

$$|\mathscr{U}_{\mathscr{W}'}| = |\mathscr{U}| + 2$$

Corollary 7.5. *Let W be of type II with $4 < a = m_{st} < \infty$ and $4 < b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 26 + 2(a-5) + 2(b-5) = 6 + 2(a+b)$$

7.5. **Type III.** We end this work with a complete description of the ultra-low elements for type III. Type III is very similar to type II, as illustrated in Figure 9.

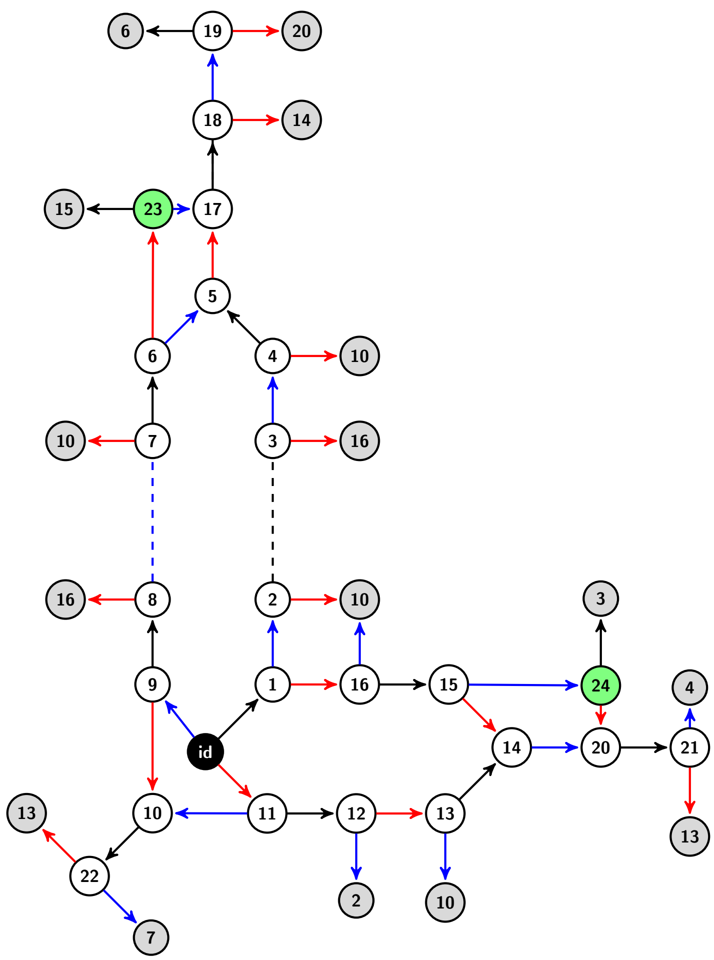


FIGURE 9. The automaton $\mathcal{A}_o$ for (W,S) of Type III. All transitions and ultra-low elements remain the same as in type II, with the exception of the states at the "top" of the diagram (compare the top part of Figure 8 with this automaton to see that there is an "extra" non-spherical state (labelled 18) here.

Specifically, the following results from type II carry over to type III with of course, $a = 4$ being even. The proofs are exactly the same, and so we leave the computations to the reader. These results correspond to all the same transitions in Figure 8 as in Figure 9 with the exception of the "extra" state and it's transitions at the top of the diagram (transitions $18 \to 19, 19 \to 6, 19 \to 20$).

(i) Lemma 7.17 (corresponding to the transition $10 \to 23$ in Figure 9).
(ii) Lemma 7.18 (for the transitions $5 \to 17, 17 \to 18, 6 \to 23, 14 \to 20, 20 \to 21$ and $15 \to 24$ in Figure 9).
(iii) Lemma 7.19 (for the transitions $2 \to 10, 16 \to 10, 7 \to 10, 13 \to 10$ and $4 \to 10$ in Figure 9).
(iv) Lemma 7.20 (for the transitions $3 \to 16, 8 \to 16, 12 \to 2$ in Figure 9).
(v) Lemma 7.21 excluding (v) (for the transitions $24 \to 3, 21 \to 4, 21 \to 13, 23 \to 15, 18 \to 14, 22 \to 13, 22 \to 7$ in Figure 9).

The following results are specific to type III and correspond to the transitions $18 \to 19, 19 \to 6$ and $19 \to 20$.

Lemma 7.22. *Let W be of type III. Then $T(w_{ut}st) \xrightarrow{u} T(w_{ut}stu)$ is a transition in $\mathcal{A}_o$ and $w_{ut}stu$ is ultra-low.*

Proof. We compute $\Phi^1(tsw_{ut}) = \{\alpha_t, s\alpha_t, t\alpha_u\}$ and using Proposition 2.2 we compute $\Phi^1(utsw_{ut}) = \{\alpha_u, us\alpha_t, ut\alpha_u\}$. Then by what was computed in Lemma 7.18 (i) we have

$$\Phi^1(utsw_{ut}) \cap \Phi(s[t,u]_{b-1}) = \{us\alpha_t\}$$

□

The next result includes the remaining transitions.

Lemma 7.23. *Let W be of type III. Then*

(i) $T(w_{ut}stu) \xrightarrow{t} T(utut)$
(ii) $T(w_{ut}stu) \xrightarrow{s} T(w_{st}u)$

are transitions in $\mathcal{A}_o$.

Proof. (i) From Lemma 7.22 we have $\Phi^1(utsw_{ut}) = \{\alpha_u, us\alpha_t, ut\alpha_u\}$ and compute $\Phi^1(tutsw_{ut}) = \{\alpha_t, ut\alpha_u, tuts\alpha_t\}$. Since $tuts\alpha_t \notin \mathscr{E}$ The result is immediate. Similarly, for (ii) we compute $\Phi^1(sutsw_{ut}) = \{\alpha_s, \alpha_u, u\alpha_t, sut(\alpha_u)\}$ and $sut(\alpha_u) \notin \mathscr{E}$ by Lemma 7.6. By Lemma 7.18 (ii) we then have $\partial T(w_{ut}stus) = \partial T(w_{st}u)$. □

This concludes the proof for W of type III.

Corollary 7.6. *Let W be of type (II) with $a = 4$ and $5 \leq b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 25 + 2(b-5) = 15 + 2b$$

8. Further results on tight gates

We conclude this note by summarising some results on tight gates which follow by of our calculations in Section 7 and Section A. The conclusion of the results in this section is Theorem 6.

Corollary 8.1. *Let W be a Coxeter group of one of the following types*

(i) W is finite dihedral,
(ii) W is irreducible rank 3
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then for each non-simple super-elementary root β there is a unique pair of tight gates x, y such that

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

and $\Phi^R(x) = \Phi^R(y) = \{\beta\}$.

Proof. (i) This follows by Lemma 7.3 (iii).
(ii) For irreducible rank 3 Coxeter groups with a linear graph, see the computations in Section A. For the rank 3 groups of affine type the results can be visually verified by the illustrations of the cone type partition $\mathscr{T}$ in Figure 4. Rank 3 groups with a complete graph are addressed in (iv).
(iii) This follows trivially, since there are no non-simple super-elementary roots in this case.
(iv) This follows by the discussion in Corollary 7.3.

□

Corollary 8.2. *Let W be a Coxeter group of a type listed in Corollary 8.1. Then*

$$|\Gamma^0| = 2|\mathscr{E}| - |S|$$

Proof. Note that in each case, we have $\mathscr{E} = \mathcal{S}$, since:

(i) When W is finite dihedral, $\mathscr{E} = \Phi^+ = \mathcal{S}$ by Lemma 7.3 (iii).
(ii) By [26, Proposition 6.0.7] for rank 3 we have $\mathscr{E} = \mathcal{S}$.
(iii) When W is right-angled, by Lemma 2.4 we have $\mathscr{E} = \mathcal{S} = \Delta$.
(iv) When Γ_W is a complete graph, then again by Lemma 2.4 if $\beta \in \mathscr{E}$ the subgraph $\Gamma(\beta)$ cannot contain a curcuit or an infinite bond, hence $\mathscr{E} = \bigcup_{\{s,t\}\subseteq S} \Phi^+_{\langle s,t\rangle}$ with $m(s,t) < \infty$. Thus again by Lemma 7.3 (iii) $\mathscr{E} = \mathcal{S}$.

Then for each simple root, the corresponding simple reflection is a tight gate and it's own witness. By Corollary 8.1 for each non-simple super elementary root there is a unique pair of tight gates x, y such that $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^R(x) = \Phi^R(y) = \{\beta\}$. Hence

$$|\Gamma^0| = 2 * (|\mathscr{E}| - |S|) + |S| = 2|\mathscr{E}| - |S|$$

□

Appendix A. Tight gates in rank 3

We utilise Algorithm 6.1 to compute and record the tight gates in rank 3 and their final roots. Retaining the labelling convention as described in Figure 6 let

$$X = \Big(W_{\langle s,t\rangle} \setminus \{e, w_{s,t}\}\Big) \cup \Big(W_{\langle t,u\rangle} \setminus \{e, w_{t,u}\}\Big)$$

where $w_{s,t}$ and $w_{t,u}$ are the longest elements of their respective dihedral groups. Let $b = m_{u,t}$. The elements X are tight gates by Corollary 7.1.

A.1. **Type I.** The tight gates are:

$$\Gamma^0 = X \cup \{uts, s[t,u]_{b-2}, us[t,u]_{b-2}, tus[t,u]_{b-2}, utus[t,u]_{b-2}, s[t,u]_{b-1}\}$$

The final roots of the tight gates in X are $\Phi^+_{W_{\langle s,t\rangle}} \cup \Phi^+_{W_{\langle t,u\rangle}}$. The final roots of the remaining tight gates are respectively:

$$\{(1,1,c_1), (c_1,c_1,c_1^2-1), (c_1,c_1,1), (c_1,c_1,1), (c_1,c_1,c_1^2-1), (1,1,c_1)\}$$

Note that these are the elementary roots with full support and that for each root there is a single pair of tight gates with the root as their final root.

A.2. **Type II.** We note that from Lemma 7.6 for types II and III there is a single elementary root of full support. The tight gates are:

$$\Gamma^0 = X \cup \{u[t,s]_{a-1}, s[t,u]_{b-1}\}$$

The final root of $u[t,s]_{a-1}$ and $s[t,u]_{b-1}$ is the root $(c_1, 1, c_2)$.

A.3. **Type III.** The tight gates are:

$$\Gamma^0 = X \cup \{utst, s[t,u]_{b-1}\}$$

The final root of $utst$ and $s[t,u]_{b-1}$ is also the root $(c_1, 1, c_2)$.

References

[1] Peter Abramenko and Kenneth Brown. *Buildings: Theory and Applications*, volume 248 of *Graduate Texts in Mathematics*. Springer, 2008.

[2] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[3] Brigette Brink and Robert Howlett. A finiteness property and an automatic structure for coxeter groups. *Math. Ann.*, 296:179–190, 1993.

[4] Brigitte Brink. The set of dominance-minimal roots. *J. Algebra*, 206(2):371–412, 1998.

[5] Nathan Chapelier-Laget and Christophe Hohlweg. Shi arrangements and low elements in affine coxeter groups. *Canad. J. Math.*, pages 1–31, 2022.

[6] Patrick Dehornoy, Matthew Dyer, and Christophe Hohlweg. Garside families in Artin-Tits monoids and low elements in Coxeter groups. *C. R. Math. Acad. Sci. Paris*, 353(5):403–408, 2015.

[7] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.4)*. `https://www.sagemath.org`, 2024.

[8] Matthew Dyer. On the weak order of coxeter groups. *Canad. J. Math.*, 71(2):299–336, 2019.

[9] Matthew Dyer, Susanna Fishel, Christophe Hohlweg, and Alice Mark. Shi arrangements and low elements in coxeter groups. *Proc. Lond. Math. Soc. (3)*, 129(2):e12624, 2024.

[10] M.J. Dyer. n-low elements and maximal rank k reflection subgroups of coxeter groups. *J. Algebra*, 607:139–180, 2022. Special Issue dedicated to P. Dehornoy.

[11] Xiang Fu. *Root systems and reflection representations of Coxeter groups*. PhD Thesis, The University of Sydney, 2011.

[12] Lorenz Gilch, Sebastian Müller, and James Parkinson. Asymptotic entropy of random walks on fuchsian buildings and kac–moody groups. *Math. Z.*, 285(3):707–738, 2017.

[13] George Grätzer and Friedrich Wehrung, editors. *Lattice Theory: Special Topics and Applications, Volume 2*. Encyclopaedia of Mathematical Sciences. Springer International Publishing, Cham, Switzerland, 2016.

[14] Christophe Hohlweg and Jean-Philippe Labbé. On inversion sets and the weak order in coxeter groups. *European J. Combin.*, 55:1 – 19, 2016.

[15] Christophe Hohlweg, Philippe Nadeau, and Nathan Williams. Automata, reduced words and garside shadows in coxeter groups. *J. Algebra*, 457:431–456, 2016.

[16] Dyer M. Hohlweg, C. Small roots, low elements, and the weak order in coxeter groups. *Advances in Math.*, 301:739–784, 2016.

[17] Nadeau P. Hohlweg, C. and N. Williams. Automata, reduced words and garside shadows in coxeter groups. *J. Algebra (computational section)*, 457:331–456, 2016.

[18] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[19] Damian Osajda and Piotr Przytycki. Coxeter groups are biautomatic, (Preprint) 2022. `https://arxiv.org/abs/2206.07804`.

[20] James Parkinson and Yeeka Yau. Coxeter systems for which the brink-howlett automaton is minimal. *J. Algebra*, 527:437–446, 2019.

[21] James Parkinson and Yeeka Yau. Cone types, automata, and regular partitions in coxeter groups. *Adv. Math.*, 398:108146, 2022.

[22] Piotr Przytyck and Yeeka Yau. A pair of Garside shadows. *Algebr. Comb.*, 7(6):1879–1885, 2024.

[23] Mark Ronan. *Lectures on buildings*. University of Chicago Press, Chicago, IL. Updated and revised.

[24] Fabricio Dos Santos. Garside shadows and biautomatic structures in coxeter groups, (Preprint) 2025. `https://arxiv.org/abs/2505.21718`.

[25] Jian-Yi Shi. Sign types corresponding to an affine weyl group. *J. Lond. Math. Soc. (2)*, s2-35(1):56–74, 1987.

[26] Yeeka Yau. *Automatic Structures for Coxeter Groups*. PhD Thesis, University of Sydney, 2021.

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